2024 4G3 Additional Mathematics Preliminary Examinations Marking Scheme

Section A (41 marks)

1	A man bought a new car. The value of the car depreciated with time so that its value, $\$P$, after t months' use is given by $P = 175\ 000e^{-kt}$, where k is a constant.	
	(a)	Find the value of the car, $\$P$, when the man bought it. [1] When $t = 0$, $P = 175000e^{-k(0)}$, $P = 175000$ [B1]
	The value of the car is expected to be $\$162\ 000$ after eight months' use.	
	(b)	Show that $k = 0.01$. [2] When $t = 8$, $175000e^{-8t} = 162000$ [M1] $e^{-8k} = \frac{162000}{175000}$ $k = -\frac{1}{8}\ln\frac{162000}{175000}$ $k = 0.0096487$ $k = 0.01$ [A1]
	(c)	Use the result from **part (b)** to determine the age of the car correct to the nearest month, when its value reached half of the original value when the man bought it. [2] $175000e^{-0.01t} = \frac{175000}{2}$ [M1] $e^{-0.01t} = \frac{1}{2}$ $-0.01t = \ln\frac{1}{2}$ $t = \frac{1}{-0.01}\ln\frac{1}{2}$ $t = 69.31$ $t = 70$ months [A1]

2	**A calculator must not be used in this question.**	
	(a)	Show that $\cot 15° = \sqrt{3}+2$. [4]

LHS $= \cot 15°$

$= \dfrac{1}{\tan(60°-45°)}$

$= 1 \div \left[\dfrac{\tan 60° - \tan 45°}{1+\tan 60° \tan 45°}\right]$ [M1 – application of additional formula]

$= 1 \div \left[\dfrac{\sqrt{3}-1}{1+(\sqrt{3})(1)}\right]$ [M1 – exact values of trigonometric functions for special angles]

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1}$

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$ [M1 – multiplication by conjugate]

$= \dfrac{1+2\sqrt{3}+3}{(\sqrt{3})^2-(1)^2}$

$= \dfrac{2\sqrt{3}+4}{2}$ [A1]

$= \sqrt{3}+2$

$=$ RHS (shown)

OR

$\cot 15$

$= \dfrac{\cos 15}{\sin 15}$

$= \dfrac{\cos 60 \cos 45 + \sin 60 \sin 45}{\sin 60 \cos 45 - \cos 60 \sin 45}[M1]$

$= \dfrac{\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}}[M1]$

$= \dfrac{\frac{1+\sqrt{3}}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1}\times\dfrac{1+\sqrt{3}}{1+\sqrt{3}}[M1]$

$= \dfrac{3+2\sqrt{3}+1}{3-1}$

$= 2+\sqrt{3}[A1]$

OR

$\cot 15$

$= \dfrac{1}{\tan(45-30)}$

$= \dfrac{1+\tan 45 \tan 30}{\tan 45 - \tan 30}[M1]$

$= \dfrac{1+1\times\dfrac{1}{\sqrt{3}}}{1-\dfrac{1}{\sqrt{3}}}[M1]$

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1}\times\dfrac{1+\sqrt{3}}{1+\sqrt{3}}[M1]$

$= \dfrac{3+2\sqrt{3}+1}{3-1}$

$= 2+\sqrt{3}[A1]$

OR

$\cot 15$

$= \dfrac{\cos 15}{\sin 15}$

$= \dfrac{\cos 45 \cos 30 + \sin 45 \sin 30}{\sin 45 \cos 30 - \cos 45 \sin 30}[M1]$

$= \dfrac{\dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2}+\dfrac{1}{\sqrt{2}}\times\dfrac{1}{2}}{\dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2}+\dfrac{1}{\sqrt{2}}\times\dfrac{1}{2}}[M1]$

$= \dfrac{\dfrac{1+\sqrt{3}}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}}$

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1}\times\dfrac{1+\sqrt{3}}{1+\sqrt{3}}[M1]$

$= \dfrac{3+2\sqrt{3}+1}{3-1}$

$= 2+\sqrt{3}[A1]$

	(b)	Use the result from **part (a)** to find an expression for $\operatorname{cosec}^2 15°$, in the form $p+q\sqrt{3}$ where p and q are integers. [2] $\operatorname{cosec}^2 15° = 1+\cot^2 15°$ [M1 – application of special identities] $= 1+\left(\sqrt{3}+2\right)^2$ $= 1+3+4\sqrt{3}+4$ $= 8+4\sqrt{3}$ [A1] $= 4\left(2+\sqrt{3}\right)$ OR $\operatorname{cosec}^2 15°$ $= \dfrac{1}{\sin^2 15°}$ $= \cot 15° \times \dfrac{1}{\sin 15°} \div \cos 15°$ $= \left(\sqrt{3}+2\right) \times \dfrac{1}{\sin 15° \cos 15°}$ $= \left(\sqrt{3}+2\right) \times \dfrac{1}{\sin(45\text{-}30)° \cos(45-30)°}$ $= \left(\sqrt{3}+2\right) \times \dfrac{1}{\dfrac{\sqrt{2}\sqrt{3}-\sqrt{2}}{4} \times \dfrac{\sqrt{2}\sqrt{3}+\sqrt{2}}{4}}$ $[M1 -$ app of correctly formed special identities$]$ $= \dfrac{\sqrt{3}+2}{\dfrac{6-2}{10}}$ $= \left(\sqrt{3}+2\right) \times \dfrac{16}{4}$ $= 4\sqrt{3}+8 [A1]$
3	Given that $\int_0^m \left(2e^{2x} - \dfrac{5}{2}e^{-2x}\right) \mathrm{d}x = \dfrac{3}{4}$, where m is a positive constant,	
	(a)	show that $4e^{4m} - 12e^{2m} + 5 = 0$. [3] $\int_0^m \left(2e^{2x} - \dfrac{5}{2}e^{-2x}\right) \mathrm{d}x = \dfrac{3}{4}$ $\left[e^{2x} + \dfrac{5e^{-2x}}{4}\right]_0^m = \dfrac{3}{4}$ [M1 – integration] $\left[e^{2m} + \dfrac{5e^{-2m}}{4}\right] - \left[1+\dfrac{5}{4}\right] = \dfrac{3}{4}$ [M1 – substitution of limits] $e^{2m} + \dfrac{5}{4e^{2m}} - 3 = 0$ [A1] $4e^{4m} - 12e^{2m} + 5 = 0$ (shown)

	(b)	Use the result from **part (a)** and a suitable substitution to find the value of m.	[4]
		$4e^{4m}-12e^{2m}+5=0$ Let $u=e^{2m}$, $4u^2-12u+5=0$ [M1 – substitution] $(2u-5)(2u-1)=0$ [M1 – factorisation of quadratic equation] $u=\frac{5}{2}$ or $u=\frac{1}{2}$ $e^{2m}=\frac{5}{2}$ or $e^{2m}=\frac{1}{2}$ $2m=\ln\frac{5}{2}$ or $2m=\ln\frac{1}{2}$ [M1 – application of ln] $m=\frac{1}{2}\ln\frac{5}{2}$ or $m=\frac{1}{2}\ln\frac{1}{2}$ $m=0.458$ or $m=-0.347$ (reject) Therefore, $m=0.458$. [A1 – includes rejecting $m=-0.347$]	
4	The point A lies on the curve $y=x\ln x$. The tangent to the curve at A is parallel to the line $y=2x+3$.		
	(a)	Find the exact coordinates of A.	[4]
		$m_A=2$ $\frac{dy}{dx}=2$ [M1] Given $y=2x\ln x$, $\frac{dy}{dx}=x\left(\frac{1}{x}\right)+\ln x$ [M1 – differentiation using product rule] $\frac{dy}{dx}=1+\ln x$ $1+\ln x=2$ $\ln x=1$ $x=e$ [M1] When $x=e$, $y=e\ln e$ $y=e$ $A(e,e)$ [A1]	

	The normal to the curve $y = x\ln x$ at A meets the line $y = 2x+3$ at the point B.	
	(b)	Show that the x-coordinate of B is $k(e-2)$, where k is a constant to be found. [3] Let equation of normal at A be $y = mx + c$. $m = -\frac{1}{2}$, $A(e,e)$ $y - e = -\frac{1}{2}(x-e)$ [M1] $y = -\frac{1}{2}x + \frac{1}{2}e + e$ $y = -\frac{1}{2}x + \frac{3}{2}e$ - eq (1) $y = 2x+3$ - eq (2) $(1) = (2)$ $-\frac{1}{2}x + \frac{3}{2}e = 2x+3$ [M1] $\frac{5}{2}x = \frac{3}{2}e - 3$ $x = \frac{3}{5}e - \frac{6}{5}$ $x = \frac{3}{5}(e-2)$ (shown) [A1]
5	**(a)**	Prove the identity $\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x} = \tan x$. [3] LHS $= \frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}$ $= \frac{1-(1-2\sin^2 x)+2\sin x\cos x}{1+(2\cos^2 x-1)+2\sin x\cos x}$ [M1 – application of trigonometric identity where $2x$ is removed and this step allows for factorization next] $= \frac{1-1+2\sin^2 x+2\sin x\cos x}{1+2\cos^2 x-1+2\sin x\cos x}$ $= \frac{2\sin^2 x+2\sin x\cos x}{2\cos^2 x+2\sin x\cos x}$ $= \frac{2\sin x(\sin x+\cos x)}{2\cos x(\sin x+\cos x)}$ [M1 – factorisation and simplification] $= \frac{\sin x}{\cos x}$ [A1] $= \tan x$ OR

		LHS $= \dfrac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}$ $= \dfrac{1-\left(\cos^2 x-\sin^2 x\right)+2\sin x\cos x}{1+\left(\cos^2 x-\sin^2 x\right)+2\sin x\cos x}$ $= \dfrac{\cos^2 x+\sin^2 x-\cos^2 x-\sin^2 x+2\sin x\cos x}{\cos^2 x+\sin^2 x+\cos^2 x-\sin^2 x+2\sin x\cos x}$ [M1 – application of trigonometric identity where 2x is removed and this step allows for factorization next] $= \dfrac{2\sin^2 x+2\sin x\cos x}{2\cos^2 x+2\sin x\cos x}$ $= \dfrac{2\sin x\left(\sin x+\cos x\right)}{2\cos x\left(\sin x+\cos x\right)}$ [M1 – factorisation and simplification] $= \dfrac{\sin x}{\cos x}$ [A1] $= \tan x$
	(b)	Hence solve the equation $\dfrac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x} = 3\cot 2x$ for $0° < x < 180°$. [4] $\dfrac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x} = 3\cot 2x$ $\tan x = 3\cot 2x$ $\tan x = \dfrac{3}{\tan 2x}$ $\tan x = \dfrac{3}{\frac{2\tan x}{1-\tan^2 x}}$ [M1 – application of double angle formula] $\tan x = \dfrac{3\left(1-\tan^2 x\right)}{2\tan x}$ $2\tan^2 x = 3-3\tan^2 x$ $5\tan^2 x = 3$ $\tan^2 x = \dfrac{3}{5}$ $\tan x = \pm\sqrt{\dfrac{3}{5}}$ [M1 – forming trigonometric equation] Range: $0° < x < 180°$. x is in all quadrants 1 and 2. Reference angle $= \tan^{-1}\sqrt{\dfrac{3}{5}}$ [M1 – reference angle] $= 37.76124°$ $x = 37.8°$ and $x = 180° - 37.76124°$ $x = 142.2°$ [A1 – for both answers]

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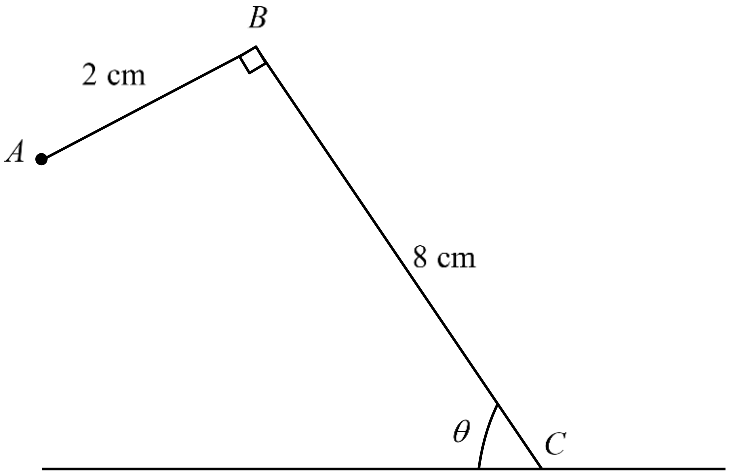
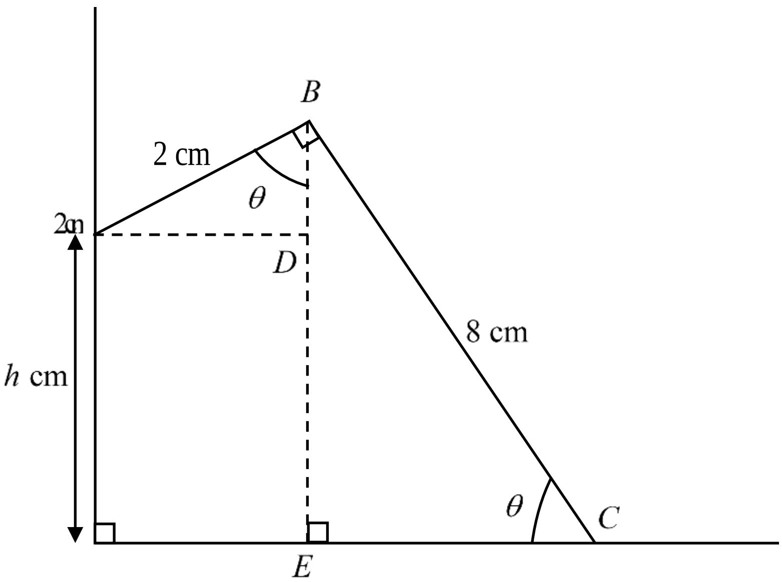


The diagram shows two rods AB and BC rigidly hinged at B so that angle $ABC = 90°$. The lengths of AB and BC are 2 cm and 8 cm respectively. The point C is fixed on horizontal ground and the rod BC rotates in a vertical plane with the rod BC inclined at an angle θ to the ground.

(a) Show that the height, h cm, of A above the ground is given by $h = a\sin\theta - b\cos\theta$, where a and b are integers to be found. [2]



$BD = 2\cos\theta$

$BE = 8\sin\theta$ [M1 – both BD and BE]

$h = 8\sin\theta - 2\cos\theta$ [A1]

	(b)	Using the values of a and b found in **part (a)**, express h in the form $R\sin(\theta-\alpha)$, where $R>0$ and $0\le\alpha\le\frac{\pi}{2}$. [4] $8\sin\theta-2\cos\theta \quad =R\sin(\theta-\alpha)$ $=R\sin\theta\cos\alpha-R\cos\theta\sin\alpha$ $R\cos\alpha=8$ - eq (1) $R\sin\alpha=2$ - eq (2) [M1 – application of addition formula and forming 2 equations] $R=\sqrt{8^2+2^2}$ [M1 – R] $R=\sqrt{68}$ $R=2\sqrt{17}\,/\,8.25$ $\tan\alpha=\frac{2}{8}$ [M1 – $\tan\alpha$] $\alpha=0.24498$ rad Therefore $8\sin\theta-2\cos\theta \quad =2\sqrt{17}\sin(\theta-0.245)$ [A1]
	(c)	Hence, state the maximum value of h and find the corresponding value of θ. [3] $h=2\sqrt{17}\sin(\theta-0.24498)$ $h_{\max}=2\sqrt{17}\,/\,8.25$ cm [B1 – $h_{\max}$] At maximum, $\sin(\theta-0.24498)=1$ [M1 – $\sin(\theta-0.24498)=1$] $(\theta-0.24498)=\frac{\pi}{2}$ $\theta=1.82$ rad [A1]

Name: ______________________________ (   )

Class: ________________________________

Questions	7	8	9	10	11	12
Marks						

Section B (49 marks)

7	The cubic polynomial $\mathrm{f}(x)$ is such that the coefficient of x^3 is 1 and the roots of $\mathrm{f}(x)=0$ are m, $2m$ and $(1-m)$, where $m>0$. It is given that $\mathrm{f}(x)$ has a remainder of 30 when divided by $1-x$.	
	(a)	Show that $2m^3-3m^2+m-30=0$. [3] Given roots, $\mathrm{f}(x)=(x-m)(x-2m)(x-1+m)$ [M1] $\mathrm{f}(x)=(x^2-2mx-mx+2m^2)(x-1+m)$ $=(x^2-3mx+2m^2)(x-1+m)$ $=x^3-x^2+mx^2-3mx^2+3mx-3m^2x+2m^2x-2m^2+2m^3$ [M1 – expansion] Given, $\mathrm{f}(1)=30$, $1-1+m-3m+3m-3m^2+2m^2-2m^2+2m^3=30$ $2m^3-3m^2+m-30=0$. (shown) [A1]
	(b)	Hence, find a value for m and show that there are no other real values of m which satisfy this equation. [4] Let $\mathrm{g}(m)=2m^3-3m^2+m-30$. $\mathrm{g}(3)=2(3)^3-3(3)^2+3-30=0$ Hence, $(m-3)$ is a factor of $\mathrm{g}(m)$. [M1 – first factor] $2m^3-3m^2+m-30=(m-3)(2m^2+km+10)$ Equating coefficient of m^2, $-3=k-6$ $k=3$ Therefore, $2m^3-3m^2+m-30=(m-3)(2m^2+3m+10)$ [M1 – quadratic factor $(2m^2+3m+10)$] For $(2m^2+3m+10)$,

		<u>Method 1</u> $b^2-4ac=3^2-4(2)(10)$ [M1 – discriminant] $b^2-4ac=-71<0$ Since $b^2-4ac<0$, $2m^2+3m+10=0$ has no solution. <u>Method 2</u> $m=\dfrac{-3\pm\sqrt{3^2-4(2)(10)}}{2(2)}$ [OR M1 – quadratic formula] $m=\dfrac{-3\pm\sqrt{-71}}{4}$ There is no solution. Solving $2m^2+3m+10=0$, $m=3$ is the only solution and hence $2m^2+3m+10=0$ has one real root, $m=3$. [A1]
8	It is given that $y=\dfrac{x-1}{\sqrt{1+x}}$.	
	(a)	Show that $\dfrac{dy}{dx}$ can be written in the form $\dfrac{x+3}{p\sqrt{(1+x)^3}}$, where p is a constant. [4] $y=\dfrac{x-1}{\sqrt{1+x}}$ $\dfrac{dy}{dx}=\dfrac{\left(\sqrt{1+x}\right)(1)-(x-1)\left(\frac{1}{2}\right)(1+x)^{-\frac{1}{2}}}{\left(\sqrt{1+x}\right)^2}$ [M1 – application of quotient law for differentiation] [M1 – correct terms] $=\dfrac{\frac{\sqrt{1+x}}{1}-\left(\frac{x-1}{2\sqrt{1+x}}\right)}{(1+x)}$ $=\dfrac{\frac{2(1+x)-(x-1)}{2\sqrt{1+x}}}{(1+x)}$ [M1 – Simplification] $=\dfrac{2+2x-x+1}{2\sqrt{(1+x)^3}}$ $=\dfrac{x+3}{2\sqrt{(1+x)^3}}$ [A1]

	(b)	Given that y is changing at a consant rate of 0.6 units per second, find the rate of change of x when $x=3$. [2] $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ $0.6 = \frac{x+3}{2\sqrt{(1+x)^3}} \times \frac{dx}{dt}$ [M1 – connected rates of change] $\frac{dx}{dt} = 0.6 \div \frac{x+3}{2\sqrt{(1+x)^3}}$ $= 0.6 \times \frac{2\sqrt{(1+x)^3}}{x+3}$ $= \frac{6\sqrt{(1+x)^3}}{5(x+3)}$ When $x=3$, $\frac{dx}{dt} = \frac{6\sqrt{(1+3)^3}}{5(3+3)}$ $=1.6$ units per second. [A1]
	(c)	Use the result from **part (a)** to evaluate $\int_0^3 \frac{x+3}{3\sqrt{(1+x)^3}}\,dx$. [4] $y = \frac{x-1}{\sqrt{1+x}}$ $\frac{dy}{dx} = \frac{x+3}{2\sqrt{(1+x)^3}}$ $\int_0^3 \frac{x+3}{2\sqrt{(1+x)^3}}\,dx = \left[\frac{x-1}{\sqrt{1+x}}\right]_0^3$ [M1 – reverse of integration] $\int_0^3 \frac{x+3}{\sqrt{(1+x)^3}}\,dx = 2\left[\frac{x-1}{\sqrt{1+x}}\right]_0^3$ $3\int_0^3 \frac{x+3}{3\sqrt{(1+x)^3}}\,dx = 2\left[\frac{x-1}{\sqrt{1+x}}\right]_0^3$ $\int_0^3 \frac{x+3}{3\sqrt{(1+x)^3}}\,dx = \frac{2}{3}\left[\frac{x-1}{\sqrt{1+x}}\right]_0^3$ [M1 – correct relationship] $= \frac{2}{3}\left[\left(\frac{3-1}{\sqrt{1+3}}\right) - \frac{0-1}{\sqrt{1+0}}\right]$ [M1 – application of limits] $= \frac{2}{3}[1+1]$ $= \frac{4}{3}$ [A1]

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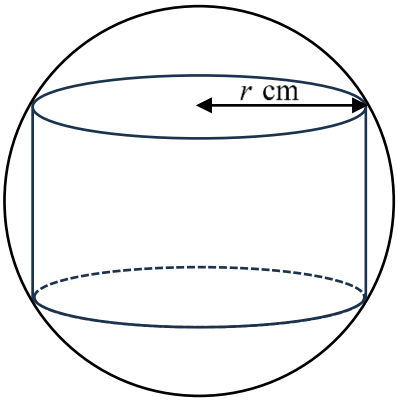
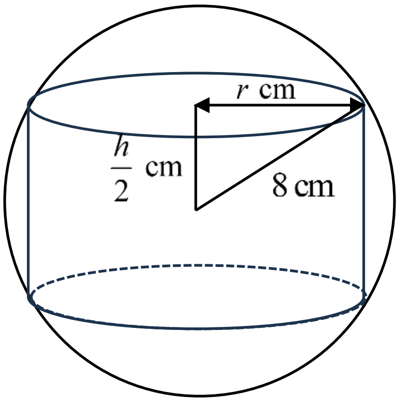


The diagram shows a cylinder of radius r cm inscribed in a sphere with a fixed internal radius of 8 cm.

(a) Given that the curved surface area of the cylinder is S cm^2, show that

$$\frac{dS}{dr} = \frac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}.$$ [5]

Let the height of the cylinder be h.



By pythagoras theorem,

$$\left(\frac{h}{2}\right)^2 + r^2 = 8^2$$

$$\frac{h^2}{4} + r^2 = 64$$ [M1 – application of pythagoras theorem]

Because $h > 0$,

$$h = \sqrt{4\left(64-r^2\right)}$$

$$h = 2\sqrt{\left(64-r^2\right)}$$

$$S = 2\pi rh$$

$$S = 4\pi r\sqrt{64-r^2}$$ [M1 – forming $S = 4\pi r\sqrt{64-r^2}$]

$$\frac{dS}{dr} = 4\pi\left[r\left(\frac{1}{2}\right)\left(64-r^2\right)^{-\frac{1}{2}}(-2r)+\left(64-r^2\right)^{\frac{1}{2}}\right]$$ [M1 – differentiating using product rule]

		$= 4\pi\left[\dfrac{-r^2}{\left(64-r^2\right)^{\frac{1}{2}}} + \dfrac{\left(64-r^2\right)^{\frac{1}{2}}}{1}\right]$ [M1 – Simplification] $= 4\pi\left(\dfrac{-r^2+64-r^2}{\left(64-r^2\right)^{\frac{1}{2}}}\right)$ $= 4\pi\left(\dfrac{-2r^2+64}{\left(64-r^2\right)^{\frac{1}{2}}}\right)$ [A1] $= \dfrac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}$ (shown)
	(b)	Given that r varies, show that S has a stationary value when the height of the cylinder is equal to twice the radius of the cylinder. [3] **<u>Method 1</u>** For S to have a stationary value, $\dfrac{\mathrm{d}S}{\mathrm{d}r} = 0.$ $\dfrac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}} = 0$ [M1 – $\dfrac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}} = 0$] $8\pi\left(32-r^2\right) = 0$ $r^2 = 32$ $r > 0,$ hence $r = \sqrt{32} = 4\sqrt{2}$ [M1 – finding r] When $r = \sqrt{32} = 4\sqrt{2}.$ $h = 2\sqrt{\left(64-\left(4\sqrt{2}\right)^2\right)}$ $h = 2\sqrt{32} = 8\sqrt{2}$ $h = 2r$ [A1] Hence, the value of S has a stationary value when the height of the cylinder is equal to its diameter. **<u>Method 2</u>** Given $h = 2r,$ $2\sqrt{\left(64-r^2\right)} = 2r$ $\sqrt{\left(64-r^2\right)} = r$ $64 - r^2 = r^2$ $2r^2 = 64$ $r^2 = 32$ $r = \sqrt{32}$ ($r > 0$) [M1 – finding r]

$$\frac{dS}{dr} = \frac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}$$

When $r=\sqrt{32}$,

$$\frac{dS}{dr} = \frac{8\pi\left(32-\sqrt{32}^2\right)}{\sqrt{64-\sqrt{32}^2}}$$ [M1 – subsitution of r]

$$=0$$ [A1]

Since $\frac{dS}{dr}=0$, S has a stationary value when the height of the cylinder is twice its radius.

(c) Determine whether this value of S is a maximum or a minimum. [2]

__**Method 1**__

$$\frac{dS}{dr} = \frac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}$$

x	$\sqrt{32}^-$	$\sqrt{32}$	$\sqrt{32}^+$
Sign of $\frac{dy}{dx}$	+	0	–
Sketch of tangent			
Sketch of curve			

[M1 – first derivative test]

Therefore, at $r=\sqrt{32}$, S is a maximum. [A1]

__**Method 2**__

$$\frac{dS}{dr} = \frac{8\pi\left(32-r^2\right)}{\sqrt{64-r^2}}$$

$$\frac{d^2S}{dr^2} = \frac{\left(64-r^2\right)^{\frac{1}{2}}(8\pi)(-2r)-8\pi\left(32-r^2\right)\left(\frac{1}{2}\right)\left(64-r^2\right)^{-\frac{1}{2}}(-2r)}{64-r^2}$$

[M1 – second derivative]

$$=\frac{-16\pi r\left(64-r^2\right)^{\frac{1}{2}}+8\pi r\left(32-r^2\right)\left(64-r^2\right)^{-\frac{1}{2}}}{64-r^2}$$

When $r=4\sqrt{2}$,

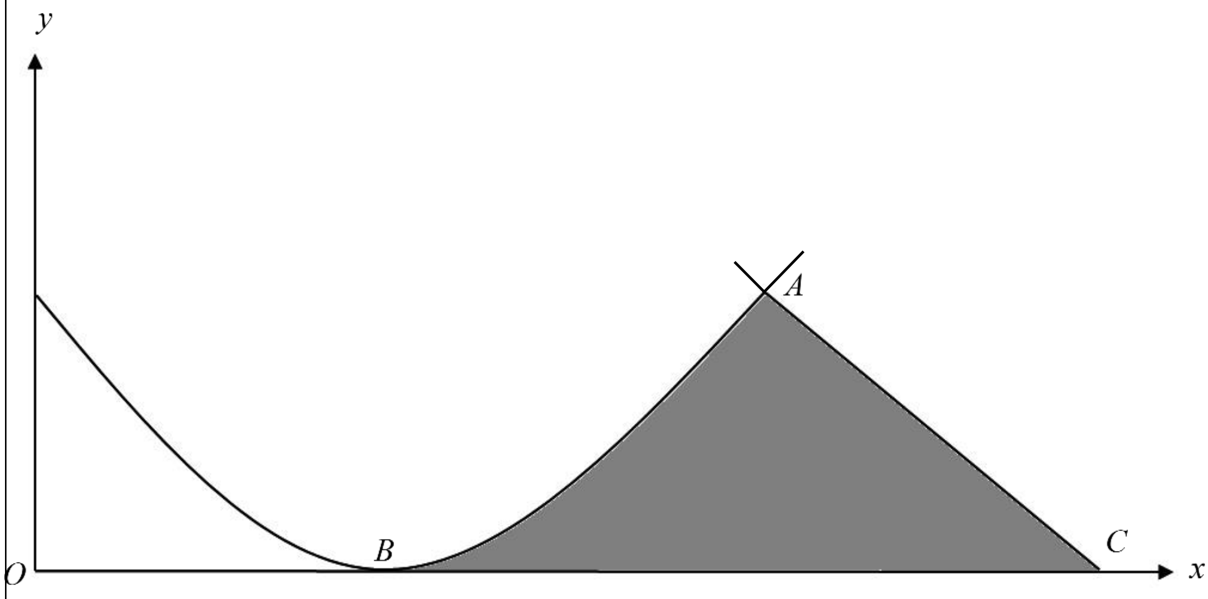
$$\frac{d^2S}{dr^2} = \frac{-16\pi\left(4\sqrt{2}\right)\left(64-\left(4\sqrt{2}\right)^2\right)^{\frac{1}{2}}+8\pi\left(4\sqrt{2}\right)\left(32-\left(4\sqrt{2}\right)^2\right)\left(64-\left(4\sqrt{2}\right)^2\right)^{-\frac{1}{2}}}{64-\left(4\sqrt{2}\right)^2}$$

$$\frac{d^2S}{dr^2} = \frac{-64\pi\sqrt{2}(32)^{\frac{1}{2}}-0}{32}$$

$$=\frac{-512\pi}{32}$$

$$=-50.3<0$$

		Since $\frac{\mathrm{d}^2S}{\mathrm{d}r^2}<0$, S is maximum at $r=4\sqrt{2}$. [A1]
10	The equation of a circle is $x^2+y^2+4x-6y-12=0$.	
	(a)	Find the radius and coordinates of the centre of the circle. [4] <u>Method 1</u> $(x-a)^2+(y-b)^2=r^2$ $x^2+y^2-2ax-2by+a^2+b^2-r^2=0$ $-2a=4$ $\quad -2b=-6$ $a=-2$ $\quad b=3$ $\quad$ [B1 – a] [B1 – b] Centre $(-2,3)$ $a^2+b^2-r^2=-12$ $(-2)^2+(3)^2-r^2=-12$ $\quad$ [M1 – substitution] $r=5$ $\quad$ [A1 – r] Method 2 $x^2+y^2+2gx+2fy+c=0$ Centre is $(-g,-f)$ Given $x^2+y^2+4x-6y-12=0$. $2g=4$ $\quad 2f=-6$ $g=2$ $\quad f=-3$ Centre $(-2,3)$ $\quad$ [B1 – a] [B1 – b] $r=\sqrt{f^2+g^2-c}$ $r=\sqrt{(-3)^2+(2)^2-(-12)}$ $\quad$ [M1 – substitution] $r=5$ $\quad$ [A1 – r]
	(b)	Find the shortest distance of the centre of the circle to the line $y=2x-3$ and hence explain whether the circle intersects the line $y=2x-3$. [6] The shortest distance is the line perpendicular to the line $y=2x-3$. Let the equation of this line be $y=mx+c$. $m=-\frac{1}{2}$ and Centre $(-2,3)$ $\quad$ [M1 – gradient $m=-\frac{1}{2}$] $y-3=-\frac{1}{2}(x+2)$ $y=-\frac{1}{2}x+2$ $\quad$ [M1 – equation of perpendicular to line] To find the point of intersection D,

		$2x-3=-\frac{1}{2}x+2$ [M1 – simultaneous equataion] $\frac{5}{2}x=5$ $x=2$ $y=2(2)-3$ $y=1$ $D(2,1)$ [B1 – coordinates of D] Distance between Centre and D $=\sqrt{(2+2)^2+(1-3)^2}$ [M1 – line segment] $=\sqrt{20}$ $=2\sqrt{5}\,/\,4.47$ Since $2\sqrt{5}\,/\,4.47<5$, the shortest distance is shorter than the radius of the circle, the circle intersects the line $y=2x-3$. [A1]
11		
	The diagram shows part of the curve $y=1-\sin x$ passing through the point A. The curve touches the x-axis at the point B. The gradient of the tangent to the curve at 2㎝ is 1 and the normal to the curve at A meet the x-axis at C.	

Show that the area of the shaded region is $\frac{1}{2}(\pi - 1)$ units^2. [12]

Let coordinates of B be $(b, 0)$.

$1 - \sin x = 0$

$\sin x = 1$

$b = x = \frac{\pi}{2}$

$B\left(\frac{\pi}{2}, 0\right)$ [B1 – coordinates of B]

Let coordinates of 2㎝ be (x, y).

$\frac{dy}{dx} = -\cos x$ [M1 – $\frac{dy}{dx}$]

$-\cos x = 1$

$\cos x = -1$

$x = \pi$

When $x = \pi$,

$y = 1 - \sin \pi$

$y = 1$

$A(\pi, 1)$. [B1 – coordinates of A]

Let equation of AC be $y = mx + c$

$m_{AC} = -1$ [M1 – gradient of normal, m_{AC}]

$A(\pi, 1)$

$y - 1 = -(x - \pi)$

$y = -x + 1 + \pi$ [M1 – equation of line AC]

Let coordinates of C be $(c, 0)$.

$-x + 1 + \pi = 0$

$x = 1 + \pi$

$C(1 + \pi, 0)$ [B1 – coordinates of C]

Area of shaded region

$= \int_{\frac{\pi}{2}}^{\pi} (1 - \sin x)\ dx + \frac{1}{2} \times [(1 + \pi) - \pi] \times 1$ [M1 – area under curve, $\int_{\frac{\pi}{2}}^{\pi} (1 - \sin x)\ dx$]

$= [x - (-\cos x)]_{\frac{\pi}{2}}^{\pi} + \frac{1}{2}$ [M1 – area of triangle $\frac{1}{2} \times [(1 + \pi) - \pi] \times 1$]

$= [x + \cos x]_{\frac{\pi}{2}}^{\pi} + \frac{1}{2}$ [M1 – integration $[x + \cos x]_{\frac{\pi}{2}}^{\pi}$]

$= \left[(\pi + \cos \pi) - \left(\frac{\pi}{2} + \cos \frac{\pi}{2}\right)\right] + \frac{1}{2}$ [M1 – substitution of limits]

	$=\left[\pi-1-\dfrac{\pi}{2}-0\right]+\dfrac{1}{2}$	[M1 – correct evaluation, including area of triangle]
	$=\dfrac{\pi}{2}-\dfrac{1}{2}$	
	$=\dfrac{1}{2}(\pi-1)$	[A1]

END OF PAPER