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PRESBYTERIAN HIGH SCHOOL

ADDITIONAL MATHEMATICS **4049/02**
Paper 2

20 August 2024 Tuesday 2 hours 15 min

2024 SECONDARY FOUR EXPRESS / FIVE NORMAL (ACADEMIC)
PRELIMINARY EXAMINATIONS

MARK SCHEME

Q1 – 8 Mr Gregory Quek
Q9 – 10 Mr Tan Lip Sing

1 **(a)** Write down, and simplify, the first three terms in the expansion of $\left(3-\frac{2}{x}\right)^5$ in descending powers of x. [2]

$$\left(3-\frac{2}{x}\right)^5 = 3^5 + 5(3)^4\left(-\frac{2}{x}\right) + \binom{5}{2}(3)^3\left(-\frac{2}{x}\right)^2 + \ldots$$

$$\left(3-\frac{2}{x}\right)^5 = 243 - \frac{810}{x} + \frac{1080}{x^2} + \ldots$$

B2: Three correct terms

(B1: Two correct terms)

(b) Given that there is no term independent of x in the expansion of $\left(5+ax^2\right)\left(3-\frac{2}{x}\right)^5$, hence find the value of the constant a. [3]

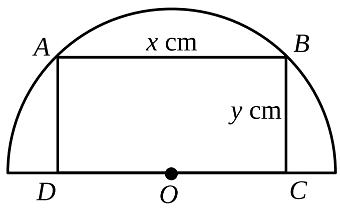
$$\left(5+ax^2\right)\left(3-\frac{2}{x}\right)^5 = \left(5+ax^2\right)\left(243-\frac{810}{x}+\frac{1080}{x^2}+\ldots\right)$$

Term independent of x = $(5)(243)+\left(ax^2\right)\left(\frac{1080}{x^2}\right)$ M1: Derive terms indep. of x

$\Rightarrow 1215 + 1080a = 0$ M1: Equate terms to zero

$\Rightarrow a = -\frac{1215}{1080} = -1.125$ A1: Accept $-1\frac{1}{8}$ or $-\frac{9}{8}$

2 In the figure, *ABCD* is a rectangle inscribed within a semicircle of radius 4 cm and centre *O*. It is given that $AB = x$ cm and $BC = y$ cm.



(a) Show that the area of the rectangle, A cm, is given by $A = \frac{1}{2}x\sqrt{64 - x^2}$. [2]

$$y^2 = 4^2 - \left(\frac{1}{2}x\right)^2$$

$$y = \sqrt{16 - \frac{1}{4}x^2}$$ M1: Correct application of Pythagoras Theorem

$$A = x\sqrt{16 - \frac{1}{4}x^2}$$

$$A = x\sqrt{\frac{1}{4}}\sqrt{64 - x^2}$$ M1: Factorise and simplify surd

$$A = \frac{1}{2}x\sqrt{64 - x^2}$$ (a.g.)

(b) Find the exact value of x for which A has a stationary value.
Give your answer in the form $k\sqrt{2}$, where k is an integer. [4]

$$\frac{dA}{dx} = \frac{1}{2}x \bullet \left[\frac{1}{2}\left(64 - x^2\right)^{-\frac{1}{2}} \bullet (-2x)\right] + \sqrt{64 - x^2} \bullet \left(\frac{1}{2}\right)$$ M1, M1: Product rule

$$\frac{dA}{dx} = -\frac{1}{2}x^2\left(64 - x^2\right)^{-\frac{1}{2}} + \frac{1}{2}\left(64 - x^2\right)^{\frac{1}{2}}$$

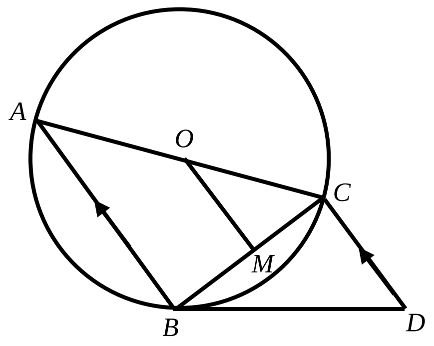
$$\frac{dA}{dx} = \frac{1}{2}\left(64 - x^2\right)^{-\frac{1}{2}}\left[-x^2 + \left(64 - x^2\right)\right] = \frac{32 - x^2}{\sqrt{64 - x^2}}$$

For stationary value, $\frac{dA}{dx} = \frac{32 - x^2}{\sqrt{64 - x^2}} = 0$ M1: Equate dA/dx to zero

$$32 - x^2 = 0$$

$$x = \sqrt{32} = 4\sqrt{2}$$ A1

3 The diagram shows a triangle *ABC* is inscribed in the circle with centre *O*.
BD is a tangent to the circle at *B* and *AB* is parallel to *CD*. Point *M* is the midpoint of *BC*.



(a) Prove that triangles *ABC* and *BCD* are similar. [3]

$\angle ABC = \angle BCD$ (*alt*. $\angle s$, $AB // CD$) M1
$\angle BAC = \angle CBD$ (*alternate segment theorem*)

Triangles ABC and BCD are similar. (*AA similarity*) A1

(b) Prove that *ABMO* is a trapezium. [2]

Since O and M are the midpoints of AC and BC respectively,
$OM // AB$ (*midpoint theorem*) M1

ABMO is a trapezium. (*one pair of parallel sides*) A1

(c) Prove that $OM = \dfrac{BC^2}{2CD}$. [3]

$\dfrac{AB}{BC} = \dfrac{BC}{CD}$ (*corr. sides of similar* Δs) M1

Since $AB = 2OM$ (*midpoint theorem*) M1

$\Rightarrow \dfrac{2OM}{BC} = \dfrac{BC}{CD}$

$\therefore OM = \dfrac{BC^2}{2CD}$ A1

***Penalise 1m per question for any missing or incorrect reasons.**

4 Milk is poured into an empty cup and heated. The temperature, T_m °C, of the milk in the cup, t minutes after it is heated, is modelled by the formula, $T_m = 5(2)^t + 20$.

(a) State the initial temperature of the milk. [1]

Initial temperature of milk = $5(2)^0 + 20 = 25$°C B1

Coffee is poured into another empty cup. The temperature, T_c °C, of the coffee in the cup, t minutes after it is poured, is modelled by the formula, $T_c = 60(2)^{-t} + 25$.

(b) Find the time taken for the temperature of the coffee to drop to 35°C. [3]

$60(2)^{-t} + 25 = 35$

$(2)^{-t} = \frac{35-25}{60} = \frac{1}{6}$ M1: Isolate $(2)^{-t}$

$\lg(2)^{-t} = \lg\left(\frac{1}{6}\right)$ M1: Take lg on both sides

$-t\lg(2) = \lg\left(\frac{1}{6}\right)$

$t = -\lg\left(\frac{1}{6}\right) \div \lg(2) = 2.5849$

$t \approx 2.58$ min ($3sf$) A1

(c) Find the time taken for the milk and the coffee to reach the same temperature. [4]

$5(2)^t + 20 = 60(2)^{-t} + 25$ M1: Equate T_m to T_c

$5(2)^{2t} + 20(2)^t = 60 + 25(2)^t$ M1: Multiply 2^t throughout / obtain quad. eqn.

$5(2)^{2t} - 5(2)^t - 60 = 0$

$(2)^{2t} - (2)^t - 12 = 0$

Let $u = (2)^t$, $u^2 - u - 12 = 0$

Let $u = (2)^t$, $5u + 20 = \frac{60}{u} + 25$
$5u^2 - 5u - 60 = 0$
$u^2 - u - 12 = 0$

$(u-4)(u+3) = 0$ M1: Solve quadratic equation

$u = 4$ *or* $u = -3$ (rejected)

$(2)^t = 4$

$t = 2$ min A1

5 It is given that $f(x) = 2x^3 - x^2 y - 13xy^2 - 6y^3$.

(a) Show that $x - 3y$ is a factor of $f(x)$. [2]

$f(3y) = 2(3y)^3 - (3y)^2 y - 13(3y)y^2 - 6$

$f(3y) = 54y^3 - 9y^3 - 39y^3 - 6y^3 = 0$ M1: Sub. into f(x) & simplify

Since $f(3y) = 0$, by Factor Theorem, $x - 3y$ is a factor of $f(x)$. AG1

(b) If y = 1, find an expression in fully factorised form for $f(x)$. [3]

Let $f(x) = 2x^3 - x^2 - 13x - 6$

$= (x-3)\left[2x^2 + bx + 2\right]$

Comparing x^2 term:

$-1 = b + (-3)(2)$

$b = 5$

M1: Comparing coefficient (or long division)

$\Rightarrow f(x) = (x-3)\left[2x^2 + 5x + 2\right]$ A1

$\Rightarrow f(x) = (x-3)(2x+1)(x+2)$ A1

(c) Hence solve the equation $2e^{6z} - e^{4z} - 13e^{2z} - 6 = 0$ and show that the solution may be written in the form $\ln\sqrt{p}$, where p is an integer. [3]

Let $x = e^{2z}$,

we get $2e^{6z} - e^{4z} - 13e^{2z} - 6 = 0$

$\Rightarrow \left(e^{2z} - 3\right)\left(2e^{2z} + 1\right)\left(e^{2z} + 2\right) = 0$ M1: Sub. $x = e^{2z}$ into (b)

$\Rightarrow e^{2z} = 3$ *or* $2e^{2z} = -1(\textit{rejected})$ *or* $e^{2z} = -2(\textit{rejected})$ A1: Seen $e^{2z} = 3$

$\ln e^{2z} = \ln 3$

$2z = \ln 3$

$z = \dfrac{1}{2}\ln 3$

$\therefore z = \ln\sqrt{3}$ A1

6 **(a)** Given that $\tan\theta = 2\operatorname{cosec}\theta$, show that $\cos^2\theta + 2\cos\theta - 1 = 0$. [3]

$\tan\theta = 2\operatorname{cosec}\theta$	
$\dfrac{\sin\theta}{\cos\theta} = \dfrac{2}{\sin\theta}$	M1: Seen either one
$\sin^2\theta = 2\cos\theta$	
$1-\cos^2\theta = 2\cos\theta$	M1: Apply Pythagorean identity
$\cos^2\theta + 2\cos\theta - 1 = 0$	AG1

(b) Using **part (a)**, find the exact value of $\cos\theta$ in simplest form, given that $0° < \theta < 90°$. [3]

$\cos^2\theta + 2\cos\theta - 1 = 0$	
$\cos\theta = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)}$	M1: Apply quadratic formula
$\cos\theta = \dfrac{-2 \pm \sqrt{8}}{2}$	
$\cos\theta = -1 \pm \sqrt{2}$	M1: Attempt to simplify
Since $0° < \theta < 90°$, $\cos\theta$ must be positive.	
$\therefore \cos\theta = -1 + \sqrt{2}$	A1

(c) Hence find the value of $\sec^2\theta$ in the form $a + b\sqrt{2}$, where a and b are integers. [5]

$\sec^2\theta = \dfrac{1}{\cos^2\theta}$	M1: Seen $\dfrac{1}{\cos^2\theta}$
$\sec^2\theta = \dfrac{1}{\left(-1+\sqrt{2}\right)^2}$	
$\sec^2\theta = \dfrac{1}{\left(\sqrt{2}\right)^2 - 2\left(\sqrt{2}\right)(1) + 1^2}$	M1: Expand the denominator
$\sec^2\theta = \dfrac{1}{3-2\sqrt{2}} \times \dfrac{3+2\sqrt{2}}{3+2\sqrt{2}}$	M1: Rationalise the denominator
$\sec^2\theta = \dfrac{3+2\sqrt{2}}{3^2 - \left(2\sqrt{2}\right)^2}$	M1: Simplify the denominator
$\sec^2\theta = 3 + 2\sqrt{2}$	A1

7 **(a)** Prove that $(\sin 2x)(\cot x) - 1 = \cos 2x$. [2]

$LHS = (\sin 2x)(\cot x) - 1$	
$= (2\sin x\cos x)\left(\dfrac{\cos x}{\sin x}\right) - 1$	M1: Seen 2sin$x$cos$x$
$= 2\cos^2 x - 1$	M1
$= \cos 2x = RHS$	(a.g)

(b) Given that $y = (\sin 2x)(\cot x) - 1$, hence show that $\dfrac{d^2y}{dx^2} + 3\left(\dfrac{dy}{dx}\right) + 2y + 9\sin 2x = 0$ may be written in the form $\tan 2x = k$, where k is a constant to be found. [4]

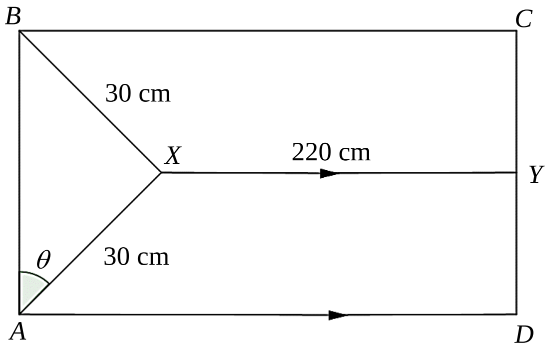
$y = (\sin 2x)(\cot x) - 1 = \cos 2x$	
$\dfrac{dy}{dx} = -2\sin 2x$	B1
$\dfrac{d^2y}{dx^2} = -4\cos 2x$	B1
$\dfrac{d^2y}{dx^2} + 3\left(\dfrac{dy}{dx}\right) + 2y + 9\sin 2x = 0$	
$-4\cos 2x + 3(-2\sin 2x) + 2\cos 2x + 9\sin 2x = 0$	M1: Correct substitution
$3\sin 2x = 2\cos 2x$	
$\tan 2x = \dfrac{2}{3}$	
$\therefore k = \dfrac{2}{3}$	A1

(c) Solve $\tan 2x = -\sqrt{3}$ for $0 \le x \le 2\pi$, giving your answers in terms of π. [4]

$\alpha = \tan^{-1}\left(\sqrt{3}\right) = \dfrac{\pi}{3}$	M1: Find reference angle
$2x = \pi - \dfrac{\pi}{3}, 2\pi - \dfrac{\pi}{3}, 3\pi - \dfrac{\pi}{3}, 4\pi - \dfrac{\pi}{3}$	M1: Find angles in 2nd and 4th quadrants
$2x = \dfrac{2\pi}{3}, \dfrac{5\pi}{3}, \dfrac{8\pi}{3}, \dfrac{11\pi}{3}$	
$x = \dfrac{\pi}{3}, \dfrac{5\pi}{6}, \dfrac{4\pi}{3}, \dfrac{11\pi}{6}$	A2: One mark for each correct pair of angles

***Penalise 1m for answers not in terms of π.**

8



The diagram shows a rectangular flag *ABCD*. *XAB* is a triangle with *AX* = *BX* = 30 cm and angle $XAB = \theta$ for $0 < \theta < 90°$. *XY* is parallel to *AD* and *XY* = 220 cm.

(a) Express the area of triangle *XAB* in the form $q\sin 2\theta$, where *q* is an integer. [2]

Area of triangle *XAB* = $\frac{1}{2}(30)(30)\sin(180° - 2\theta)$	M1: Apply formula ½*b*csin*A*
Area of triangle *XAB* = $450\sin 2\theta$	A1

(b) Given that θ can vary, find the maximum possible area of triangle *XAB* and the value of θ at which this occurs. [2]

This occurs when $\sin 2\theta = 1$,	
Maximum area of triangle *XAB* = 450 cm^2	B1: F.T.
Value of $\theta = 45°$	B1

(c) Show that the perimeter, *P* cm, of the rectangular flag *ABCD* can be expressed in the form $a\sin\theta + b\cos\theta + c$, where *a*, *b* and *c* are constants to be found. [3]

$AD = 30\sin\theta + 220$ $AB = 2 \times 30\cos\theta$	M1: Either *AD* or *AB*
Perimeter = $= 2[30\sin\theta + 220] + 2[2 \times 30\cos\theta]$	M1: Attempt to find perimeter
$P = 60\sin\theta + 120\cos\theta + 440$	A1

(d) By expressing *P* in the form $R\sin(\theta + \alpha) + c$, where *R* > 0 and $0 < \alpha < 90°$, explain if it is possible to have a flag with perimeter 550 cm. Show your working clearly. [5]

$R = \sqrt{60^2 + 120^2} = \sqrt{18000} = 60\sqrt{5}$	M1: Seen $R = \sqrt{a^2 + b^2}$
$\alpha = \tan^{-1}\left(\dfrac{120}{60}\right) = 63.434°$	M1: Seen $\alpha = \tan^{-1}\dfrac{b}{a}$
$P = 60\sqrt{5}\sin(\theta + 63.434°) + 440$	

Method 1

Let $60\sqrt{5}\sin(\theta + 63.434°) + 440 = 550$

$\sin(\theta + 63.434°) = \dfrac{550 - 440}{60\sqrt{5}}$

Reference angle = $\sin^{-1}\left(\dfrac{11}{6\sqrt{5}}\right) = 55.0739$ M1: Find reference angle

$\theta + 63.434° = 55.0739° \text{ or } 180° - 55.0739°$ M1: Find θ in 1st & 2nd quad

$\theta = -8.3601° \ (\textit{rejected}) \ \text{ or } \ 61.4921°$

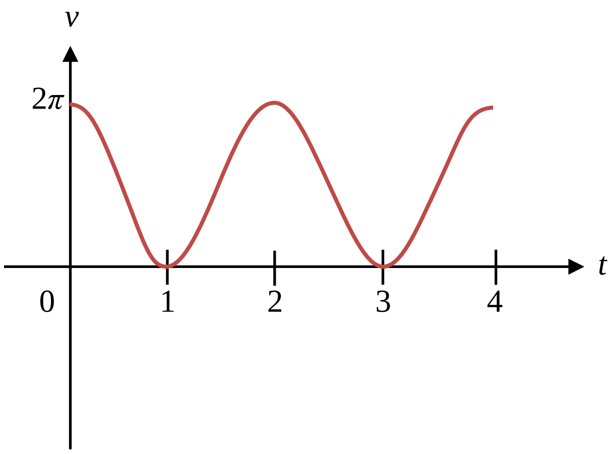
Yes, it is possible to have a flag with perimeter 550 cm when $\theta \approx 61.5°$ (1dp) A1

Method 2

$\textit{Maximum } P = 60\sqrt{5} + 440 = 574$ cm	M1
When $\theta = 90°$, $\textit{Minimum } P = 60\sqrt{5}\sin(90° + 63.434°) + 440 = 500$ cm	M1
Since $500 < P \le 574$, it is possible to have a flag with perimeter 550 cm.	A1

9 A particle moves in a straight line so that, t seconds after passing a fixed point O, its velocity, v metres per second, is given by $v = \pi\cos(\pi t) + \pi$.

(a) Sketch the velocity-time graph of the particle for $0 \le t \le 4$. [3]



B1: Correct shape   B1: Correct max & min values   B1: Correct period

(b) Determine how many times the particle is at instantaneous rest in the first 10 seconds. [1]

From the graph, the particle is at instantaneous rest when $v = 0$ at every odd second.

Therefore, there are **5 times** in the first 10 seconds. B1

(c) Explain why the particle will never return to the origin *O*. [2]

Since $v \ge 0$, the **velocity** of the particle is **never negative**, B1
hence the particle **does not change its direction of motion**. B1
Therefore, the particle will never return to the origin *O*. (a.g)

(d) Find an expression, in terms of *t*, for the displacement of the particle. [2]

$$s = \int \pi \cos(\pi t) + \pi \text{ dt}$$

$$s = \frac{\pi \sin(\pi t)}{\pi} + \pi t + c$$ M1: Apply integration

When $t = 0$, $s = 0$, thus $c = 0$.

(e) Calculate the average speed of the particle in the first 4 seconds. [3]

When $t = 0$, $s = 0$.	
When $t = 4$, $s = \sin\left(4\pi\right) + 4\pi = 4\pi$	M1: Find displacement at $t = 4$
Average speed = $\dfrac{4\pi}{4}$	M1: Find average speed
Average speed = π m/s	A1

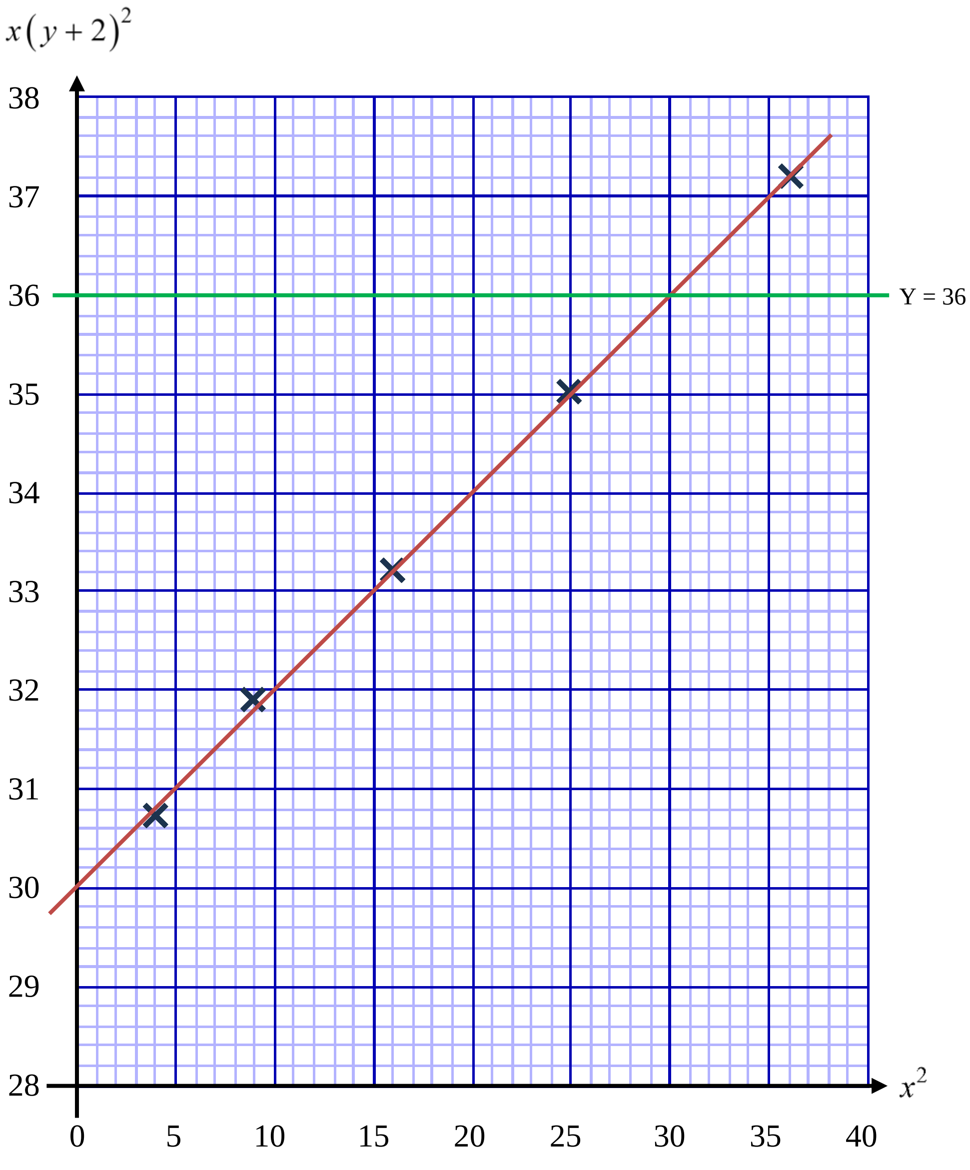
10 It is known that x and y are related by the equation $y = \sqrt{Ax + \frac{B}{x}} - 2$, where A and B are positive constants. The following table shows the values of the variables, x and y.

x	2	3	4	5	6
y	1.92	1.26	0.881	0.646	0.490

x^2	4	9	16	25	36
$x(y+2)^2$	**30.7**	**31.9**	**33.2**	**35.0**	**37.2**

(a) Plot $x(y+2)^2$ against x^2 and draw a straight line graph to illustrate the information. [3]

B2: All correct points plotted (B1: at least 3 correct) B1: Best fit line



(b) Express the equation $y = \sqrt{Ax + \frac{B}{x}} - 2$ in a form that will yield the straight line graph in **part (a)**. [2]

$y = \sqrt{Ax + \frac{B}{x}} - 2$	
$y + 2 = \sqrt{Ax + \frac{B}{x}}$	
$(y+2)^2 = Ax + \frac{B}{x}$	M1: Taking square on both sides
$x(y+2)^2 = Ax^2 + B$	A1

(c) Use your graph to estimate the value of A and of B. [2]

$A = gradient = \frac{37.2 - 33.2}{36 - 16} = 0.2$	B1: Accept +/- 0.01
$B = Y - \text{intercept} = 30$	B1: Accept +/- 0.5

(d) Explain why the graph $y = \sqrt{Ax + \frac{B}{x}} - 2$ is undefined for $x \le 0$. [2]

When $x = 0$, $\frac{B}{x}$ results in **division by zero error**.	B1
When $x < 0$, since $A > 0$ and $B > 0$, $Ax + \frac{B}{x} < 0$, hence $\sqrt{Ax + \frac{B}{x}}$ has **no real roots**.	B1
Hence, $y = \sqrt{Ax + \frac{B}{x}} - 2$ is undefined for $x \le 0$.	(a.g.)

(e) By drawing a suitable line on your graph, estimate the value of x for which $y + 2 = \frac{6}{\sqrt{x}}$.

Give your answer to 3 significant figures. [2]

$y + 2 = \frac{6}{\sqrt{x}}$	
$(y+2)^2 = \frac{36}{x}$	B1: Draw $Y = 36$ on the same axes
$x(y+2)^2 = 36$	
From the graph, when $x(y+2)^2 = 36$, $x^2 = 30$	
$\therefore x = \sqrt{30} = 5.4772 \approx 5.48\ (3sf)$	B1: Accept +/- 0.1