Name:	Index No.:	Class:

PRESBYTERIAN HIGH SCHOOL

ADDITIONAL MATHEMATICS **4049/01**
Paper 1

19 August 2024 Monday 2 hours 15 min

2024 SECONDARY FOUR EXPRESS / FIVE NORMAL (ACADEMIC)
PRELIMINARY EXAMINATIONS

MARKING SCHEME

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1 Express $\dfrac{3x^2+14x+6}{x^2(x+3)}$ as the sum of 3 partial fractions. [5]

$$\frac{3x^2+14x+6}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$ $\left[\mathbf{M1},\ \textbf{accept}\ \dfrac{\boldsymbol{Ax+B}}{\boldsymbol{x^2}} + \dfrac{\boldsymbol{C}}{\boldsymbol{x+3}}\right]$

$$3x^2+14x+6 = Ax(x+3)+B(x+3)+Cx^2$$ [**M1 for comparing numerator**]

Subst. $x=0$: $6 = 3B$ [**M1 correct method of subst. or compare coeff.**]

$$B=2$$

Subst. $x=-3$: $-9=9C$

$$C=-1$$

Subst. $x=1$: $23 = 4A+7$

$$A=4$$

Therefore, $\dfrac{3x^2+14x+6}{x^3+3x^2} = \dfrac{4}{x} + \dfrac{2}{x^2} - \dfrac{1}{x+3}$ [**A1 for 2 correct terms; A2 for all correct**]

2 **(a)** State, in terms of π,

(i) the principal value of $\tan^{-1}\left(-\sqrt{3}\right)$, [1]

(ii) the values between which the principal value of $\sin^{-1}x$ must lie. [1]

(b) Given that A is a reflex angle and $\cos A = -\frac{4}{5}$, find the exact value of $\cos(A + 30°)$ without the use of a calculator. [3]

(a) **(i)** $-\frac{\pi}{3}$ [**B1**]

(ii) $-\frac{\pi}{2} \leqslant \sin^{-1}x \leqslant \frac{\pi}{2}$ [**B1**]

(b) A lies in the 3rd quadrant.

$$\sin A = -\frac{3}{5}$$ [**B1**]

$$\cos(A + 30°) = \cos A \cos 30° - \sin A \sin 30°$$

$$= \left(-\frac{4}{5}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(-\frac{3}{5}\right)\left(\frac{1}{2}\right)$$ $\left[\textbf{M1 for } \cos 30° = \frac{\sqrt{3}}{2} \textbf{ or } \sin 30° = \frac{1}{2}\right]$

$$= \frac{3}{10} - \frac{2}{5}\sqrt{3}$$ [**A1**]

3 **(a)** Find the set of values of the constant k for which the curve $y = -kx^2 - 2x + 2k - 3$ does not intersect the x-axis. [3]

$b^2 - 4ac < 0$

$(-2)^2 - 4(-k)(2k-3) < 0$ $\left[\begin{array}{c}\textbf{M1 for } b^2 - 4ac < 0 \textbf{ and} \\ \textbf{at least 2 correct substitution of a, b, c}\end{array}\right]$

$4 + 8k^2 - 12k < 0$

$2k^2 - 3k + 1 < 0$

$(2k-1)(k-1) < 0$ [**M1 for factorisation/finding** x **– intercepts**]

$\frac{1}{2} < k < 1$ [**A1**]

(b) Using the answer in **part (a)**, explain whether it is possible for $-kx^2 - 2x + 2k - 3$ to be positive for all x. [2]

Since $\frac{1}{2} < k < 1$, **coefficient of $x^2 = -k < 0$.** **[M1]**

The curve lies entirely below the axis, therefore it is **not possible** for $-kx^2 - 2x + 2p - 3$ to be positive for all x. **[A1]**

4 A roller coaster is being designed such that the height, h m, of a rider above the ground in a section of the roller coaster ride is given by $h = 10x - 2x^2 - 4$, where x is the horizontal distance of the rider from the starting point and $1 \le x \le 4$.

(a) Express h in the form $a + b(x + c)^2$ where a, b and c are constants to be determined. [3]

$$h = 10x - 2x^2 - 4$$

$$= -2(x^2 - 5x + 2) \qquad [\textbf{M1 factorise}]$$

$$= -2\left[\left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 2\right]$$

$$= 8.5 - 2\left(x - \frac{5}{2}\right)^2 \qquad [\textbf{B1, B1 for each term}]$$

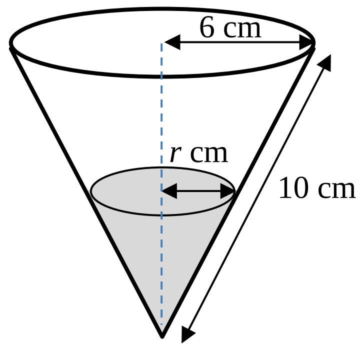
(b) Hence, explain why the rider cannot reach a height of 10 m. [1]

Since the **<u>maximum value of the rider is 8.5 m</u>** and **8.5 < 10**, the rider cannot reach a height of 10 m. **[B1]**

(c) After testing the prototype, the roller coaster designer wants to make the ride more exciting by moving the highest point of this section of the roller coaster ride up by 0.2 m and left by 0.1 m. Write down a possible new expression for h. [1]

$$h = 9.7 - 2(x - 2.4)^2 \qquad [\textbf{B1}]$$

5 Water is poured into an empty inverted conical container with radius 6 cm and slant height 10 cm. After t seconds, the radius of the top surface of the water is r cm.



(a) Show that the surface area, S, of water in contact with the container at any time is given by $S = \frac{5}{3}\pi r^2$. [2]

By similar triangles, $\frac{l}{10} = \frac{r}{6}$

$$l = \frac{5}{3}r \qquad [\mathbf{M1}]$$

$$S = \pi(r)\left(\frac{5}{3}r\right) = \frac{5}{3}\pi r^2 \qquad [\mathbf{A1}]$$

(b) Water is poured into the container such that r increases at a constant rate. Given that it takes 30 seconds to completely fill up the empty container, write down the rate at which r increases. [1]

$$\frac{dr}{dt} = \frac{6}{30} = \frac{1}{5} \text{ cm/s} \qquad [\mathbf{B1}]$$

(c) Hence calculate the rate at which S increases when the container is one-eighth filled. [4]

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt} \qquad [\mathbf{M1\ for\ chain\ rule}]$$

$$= \frac{10}{3}\pi r \times \frac{1}{5} \qquad \left[\mathbf{B1\ for\ \frac{10}{3}\pi r}\right]$$

When the container is one-eighth filled,

$$\left(\frac{r}{6}\right)^3 = \frac{1}{8}$$

$$r = \frac{1}{2} \times 6 = 3 \qquad [\mathbf{B1\ for\ r = 3}]$$

When $r = 3$, $\frac{dS}{dt} = \frac{2}{3}\pi(3) = 2\pi$ (or 6.28 cm/s to 3 s.f.) [**A1**]

6 A curve has equation $y = \dfrac{30(2x-1)^3}{\mathrm{e}^{3x}}$, where $x > 0$.

(a) Find the x-coordinates of the stationary points of the curve. [5]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^{3x} \cdot 30(3)(2x-1)^2(2) - 30(2x-1)^3 \cdot 3e^{3x}}{e^{6x}}$$

$\left[\begin{array}{c}\textbf{M1 for chain rule} \\ \textbf{i.e. } \mathbf{3(2x-1)^2(2)} \textbf{ or } \mathbf{3e^{3x}} \textbf{ seen} \\ \textbf{M1 for correct use of quotient} \\ \textbf{rule even w/o chain rule}\end{array}\right]$

$$= \frac{90e^{3x}(2x-1)^2(2-2x+1)}{e^{6x}}$$

$$= \frac{90(2x-1)^2(3-2x)}{e^{3x}}$$

[**M1 factorising**]

At stationary point, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

$(2x-1)^2 = 0$ or $3 - 2x = 0$ [**M1**]

$x = \dfrac{1}{2}$ or $x = \dfrac{3}{2}$ [**A1**]

(b) Determine the nature of each of the stationary points. [3]

x	$\left(\frac{1}{2}\right)^-$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^+$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	+	0	+
	/	—	/

x	$\left(\frac{3}{2}\right)^-$	$\frac{3}{2}$	$\left(\frac{3}{2}\right)^+$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	+	0	–
	/	—	\

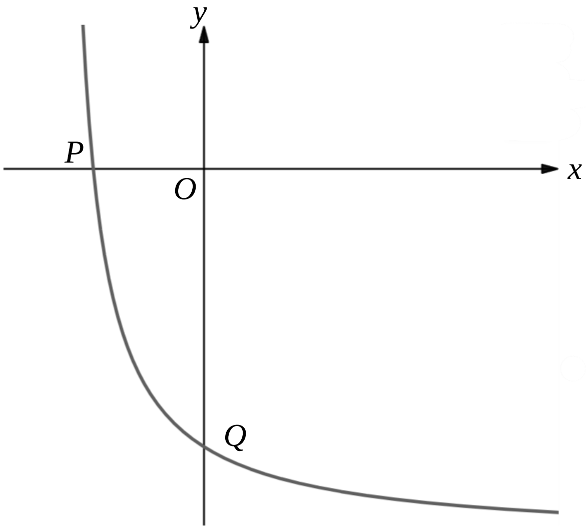
[M1 for first derivative test]

Therefore, the stationary point at $x = \dfrac{1}{2}$ is a **point of inflexion** and the stationary point at $x = \dfrac{3}{2}$ is a **maximum point**. [**A1, A1**]

(c) Using your answer in **part (b)**, infer and write down the set of values of x for which y is an increasing function. [2]

$0 < x < \dfrac{3}{2}$, $x \neq \dfrac{1}{2}$ **[B1,B1]** **OR** $0 < x < \dfrac{1}{2}$, $\dfrac{1}{2} < x < \dfrac{3}{2}$ [**B1,B1**]

7 The diagram shows part of the curve $y = \frac{40}{x+8} - 20$, where $x > -8$. The curve intersects the x-axis and y-axis at P and Q respectively. The tangent to the curve at R is parallel to line PQ.



(a) Show that the coordinates of point R are $(-4, -10)$. [5]

When $x = 0$, $y = -15$.

When $y = 0$, $x = -6$ **[B1 for either coordinates of *P* and *Q*]**

$m_{PQ} = \frac{0-(-15)}{-6-0} = -\frac{5}{2}$ **[M1]**

$\frac{dy}{dx} = -\frac{40}{(x+8)^2}$ **[B1]**

Let $-\frac{40}{(x+8)^2} = -\frac{5}{2}$

$(x+8)^2 = 16$

$x+8 = \pm 4$

$x = -12$ (reject) or -4 **[A1 for -12 and not choosing it]**

When $x = -4$, $y = -10$. Therefore $R(-4, -10)$. **[A1]**

(b) Given that the point *S* has coordinates (–2, –5), find the area of the quadrilateral *PRQS*. [2]

$$\text{Area} = \frac{1}{2}\begin{vmatrix} -6 & -4 & 0 & -2 & -6 \\ 0 & -10 & -15 & -5 & 0 \end{vmatrix} \qquad [\mathbf{M1}]$$

$$= \frac{1}{2}(60 + 60 - 30 - 30)$$

$$= 30 \text{ units}^2 \qquad [\mathbf{A1}]$$

(c) Determine, with reason, whether *PRQS* is a parallelogram. [3]

$$m_{PR} = \frac{0-(-10)}{-6-(-4)} = -5$$

$$m_{QS} = \frac{-15-(-5)}{0-(-2)} = -5$$

$$m_{RQ} = \frac{-10-(-15)}{-4-0} = -\frac{5}{4}$$

$$m_{PS} = \frac{0-(-5)}{-6-(-2)} = -\frac{5}{4} \qquad \begin{bmatrix} \textbf{M1 for 1 pair of gradient} \\ \textbf{M2 for 2 pairs of gradient} \end{bmatrix}$$

Since $m_{PR} = m_{QS}$ and $m_{RQ} = m_{PS}$, *PRQS* is a parallelogram [**A1**]

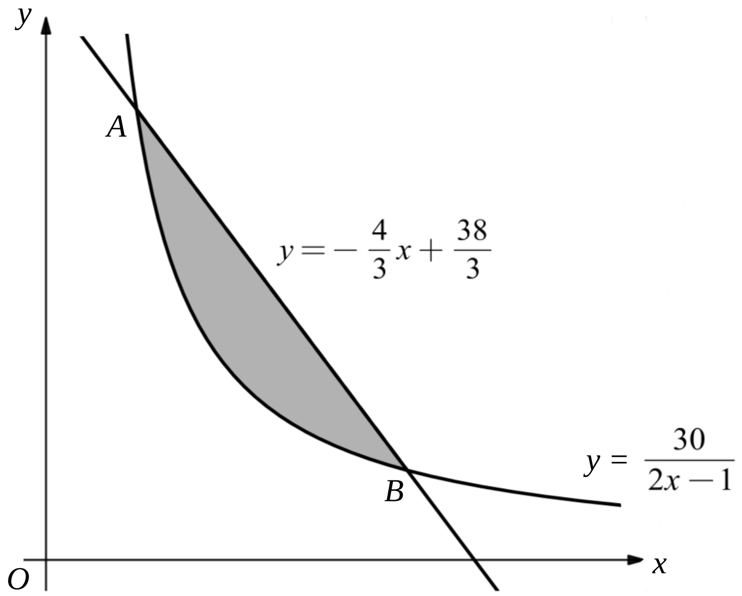
OR

Midpoint of *PQ* = $\left(\frac{-6+0}{2}, \frac{0-15}{2}\right) = (-3, -7.5)$ [**M1**]

Midpoint of *RS* = $\left(\frac{-4-2}{2}, \frac{-10-5}{2}\right) = (-3, -7.5)$ [**M1**]

Since the diagonals bisect each other, *PRQS* is a parallelogram. [**A1**]

8 The diagram shows part of the curve $y = \dfrac{30}{2x-1}$ and the line $y = -\dfrac{4}{3}x + \dfrac{38}{3}$. The curve intersects the line at the points A and B.



(a) Find the coordinates of the points A and B. [3]

$$\frac{30}{2x-1} = -\frac{4}{3}x + \frac{38}{3}$$

$$90 = (-4x+38)(2x-1) \qquad [\textbf{M1 simplify to quad func}]$$

$$90 = -8x^2 + 4x + 76x - 38$$

$$0 = 8x^2 - 80x + 128$$

$$0 = x^2 - 10x + 16$$

$$0 = (x-8)(x-2)$$

$$x = 8 \quad \text{or} \quad x = 2 \qquad [\textbf{A1}]$$

Therefore $A(2, 10)$ and $B(8, 2)$. [**A1**]

(b) Find the area of the shaded region. [4]

$$\text{Area } = \int_2^8 -\frac{4}{3}x + \frac{38}{3} - \frac{30}{2x-1}\ \mathrm{d}x \qquad [\textbf{M1 for correct order of subtraction}]$$

$$= \left[-\frac{2}{3}x^2 + \frac{38}{3}x - 15\ln(2x-1)\right]_2^8 \qquad \begin{bmatrix}\textbf{B1 for 2 terms;}\\ \textbf{B1 for 3rd term}\end{bmatrix}$$

$$= 58\frac{2}{3} - 15\ln 15 - \left(22\frac{2}{3} - 15\ln 3\right)$$

$$= 36 + 15\ln 3 - 15\ln 15$$

$$= 36 - 15\ln 5$$

$$= 11.9 \text{ units}^2 \text{ (to 3 } s.f.) \qquad [\textbf{A1}]$$

OR

$$\text{Area } = \frac{1}{2}(10+2)(6) - \int_2^8 \frac{30}{2x-1}\mathrm{d}x \qquad \begin{bmatrix}\textbf{M1 for correct integral}\\ \textbf{and subtraction}\end{bmatrix}$$

$$= 36 - [15\ln(2x-1)]_2^8 \qquad \begin{bmatrix}\textbf{B1 for 36;}\\ \textbf{B1 for integrated term}\end{bmatrix}$$

$$= 36 - (15\ln 15 - 15\ln 3)$$

$$= 11.9 \text{ units}^2 \text{ (to 3 } s.f.) \qquad [\textbf{A1}]$$

(c) Show that the area bounded by the curve, the y-axis and the lines $y = 2$ and $y = 6$ can be expressed in the form $15\ln p + q$, where p and q are constants to be determined. [3]

$$y = \frac{30}{2x-1}$$

$$2x - 1 = \frac{30}{y}$$

$$x = \frac{1}{2}\left(\frac{30}{y} + 1\right)$$

$$= \frac{15}{y} + \frac{1}{2}$$

$$\text{Area} = \int_2^6 \frac{15}{y} + \frac{1}{2}\ \mathrm{d}y \qquad [\textbf{M1 for making } \boldsymbol{x} \textbf{ the subject to form integral}]$$

$$= \left[15\ln y + \frac{y}{2}\right]_2^6 \qquad [\textbf{M1 for correct integration}]$$

$$= 15\ln 3 + 2 \qquad [\textbf{A1}]$$

9 The function f is given by $f(x) = x^3 - x^2 + ax - 1$.

It is given that $\dfrac{f(x)}{x-1} = Q(x) + \dfrac{3}{x-1}$, where $Q(x)$ is a quadratic function.

(a) Explain how you can use Remainder Theorem to show that $a = 4$. [3]

Since the **remainder is 3** when f(x) is divided by $x - 1$, [**B1**]
we let f(1) = 3 [**M1 for f(1)**]
$1 - 1 + a - 1 = 3$
$a = 4$ (shown) [**A1**]

(b) By long division, divide $f(x)$ by $x^2 + 1$.

Hence, express $\dfrac{f(x)}{x^2+1}$ in the form $px + q + \dfrac{rx}{x^2+1}$, where p, q and r are constants to be determined. [2]

$$\begin{array}{r l}
 & x - 1 \\
x^2+1 \,\big)\! & x^3 - x^2 + 4x - 1 \\
 & x^3 \qquad\;\; + x \\
\hline
 & -x^2 + 3x - 1 \\
 & -x^2 \qquad\;\; - 1 \\
\hline
 & 3x
\end{array}$$

[**M1**]

$$\frac{f(x)}{x^2+1} = x - 1 + \frac{3x}{x^2+1}$$ [**A1**]

(c) Find $\dfrac{d}{dx}\ln(x^2+1)$. [1]

$$\frac{d}{dx}\ln(x^2+1) = \frac{2x}{x^2+1}$$ [**B1**]

(d) Hence, using your results from **part (b) and (c)**, find $\displaystyle\int \frac{f(x)}{x^2+1}\,dx$. [2]

$$\int \frac{f(x)}{x^2+1}\,dx = \int x - 1 + \frac{3x}{x^2+1}\,dx$$

$$= \frac{x^2}{2} - x + \frac{3}{2}\ln(x^2+1) + c$$ [**B1 for ln term, B1 for rest of 3 terms
Allow ECF**]

10 The equation of a circle, C_1, with centre A, is $x^2 + y^2 - 6x + 8y - 75 = 0$.

(a) Find the radius of the circle and the coordinates of its centre, A. [4]

$(x-3)^2 + (y+4)^2 = 10^2$ **[M1 for $(x-3)^2$ or $(y+4)^2$ seen, M1 for 10^2]**

A (3, – 4) and radius = 10 units [**A1, A1**]

(b) A second circle, C_2, with centre B, has radius 12 units.

The equation of the perpendicular bisector of AB is $y = -\frac{1}{2}x + 5$.

Find the equation of the second circle. [5]

$m_{AB} = 2$ [**B1**]

Equation of AB is

$$y - (-4) = 2(x-3)$$

$$y = 2x - 10 \qquad [\textbf{M1}]$$

$$2x - 10 = -\frac{1}{2}x + 5$$

$$\frac{5}{2}x = 15$$

$$x = 6 \qquad [\textbf{M1}]$$

$$y = 2$$

Let centre of second circle be (a, b)

$$\left(\frac{3+a}{2}, \frac{-4+b}{2}\right) = (6, 2) \qquad [\textbf{M1}]$$

$$a = 9, \; b = 8$$

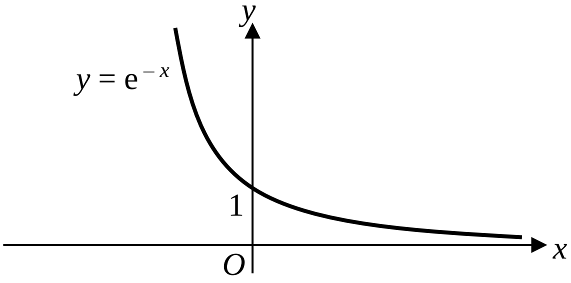
Therefore, equation of circle is $(x-9)^2 + (y-8)^2 = 144$ [**A1**]

(c) P is a point on the perpendicular bisector of AB, where P is **not** the midpoint of A and B.

Justify whether triangle PAB is an isosceles triangle. [1]

Since ***P* is equidistant from *A* and *B***, i.e. $PA = PB$, the triangle is isosceles. [**B1**]

11 **(a)** Sketch the graph of $y = \mathrm{e}^{-x}$ on the given axes. Label any axial intercepts. [1]



[**B1**]

(b) The curve $y = \mathrm{f}(x)$ is such that $\mathrm{f}'(x) = \mathrm{e}^{2x+1} + \mathrm{e}^{-x}$.

(i) Explain why the curve has no stationary points. [2]

Since $\mathrm{e}^{2x+1} > 0$ and $\mathrm{e}^{-x} > 0$ for all real values of x, [**B1**]
$\mathrm{f}'(x) \neq 0$, therefore the curve has no stationary points. [**B1**]

(ii) The curve passes through the point (ln 4, 8e). Without the use of a calculator, find an expression for $\mathrm{f}(x)$. Show all working clearly. [6]

$$\mathrm{f}(x) = \frac{\mathrm{e}^{2x+1}}{2} - \mathrm{e}^{-x} + c$$ [**B1 for 2 terms, B1 for 3rd term**]

$$8\mathrm{e} = \frac{\mathrm{e}^{2\ln 4 + 1}}{2} - \mathrm{e}^{-\ln 4} + c$$

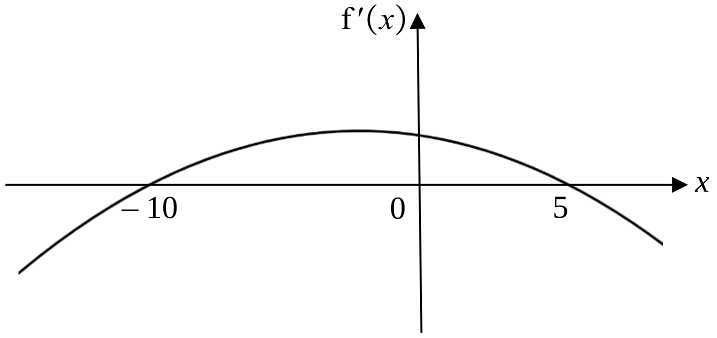
$$8\mathrm{e} = \frac{\mathrm{e}^{\ln 16 + 1}}{2} - \mathrm{e}^{\ln (4)^{-1}} + c$$ [**M1 for power law i.e. either ln 16 or ln (4)$^{-1}$ seen**]

$$8\mathrm{e} = \frac{16\mathrm{e}}{2} - \frac{1}{4} + c$$ [**M1 for $\times$e, M1 for $-\frac{1}{4}$ seen**]

$$c = \frac{1}{4}$$

Therefore, $\mathrm{f}(x) = \dfrac{\mathrm{e}^{2x+1}}{2} - \mathrm{e}^{-x} + \dfrac{1}{4}$ [**A1**]

12 The diagram shows the graph of $y = \mathbf{f'}(x)$.



(a) Write down the range of values of x for which the **function f** decreases as x increases. [1]

$x < -10$ or $x > 5$ **[B1]**

(b) The above function f is given by $f(x) = 12k^2x - 2x^3 - 3kx^2$, where k is a positive constant. By solving a suitable inequality2, use the answer in part **(a)** to find the value of the constant k. [4]

$f'(x) = 12k^2 - 6x^2 - 6kx$ [**B1**]

For f to be decreasing,

$12k^2 - 6x^2 - 6kx < 0$

$x^2 + kx - 2k^2 > 0$

$(x + 2k)(x - k) > 0$ [**M1 for correct factorisation or for $\mathbf{f'(x) < 0}$**]

$x < -2k$ or $x > k$ [**A1**]

Therefore $k = 5$ [**A1**]

(c) Explain how you can use the above graph of $y = f'(x)$ to determine the value of x for which f(x) is a maximum value. [1]

From the graph, $f'(x)$ changes from positive to negative at x =5, therefore f(x) is a maximum value when x = 5. [**B1**]

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