

Sec 4 Express A Math Prelim Paper 1 2024 Marking Scheme

1 Given that $4(5^{x+3}) = 20^{3-x}$, evaluate 10^x **without using a calculator**.

[4]

$$4(5^{x+3}) = 20^{3-x}$$

$$4(5^x)(5^3) = \frac{20^3}{20^x}$$

$$5^x(20^x) = \frac{20^3}{4(5^3)}$$

$$100^x = 16$$

$$10^x = 4$$

2 Solve the equations.

(a) $\log_2(x+4) = 2\log_2 x - 1$ [4]

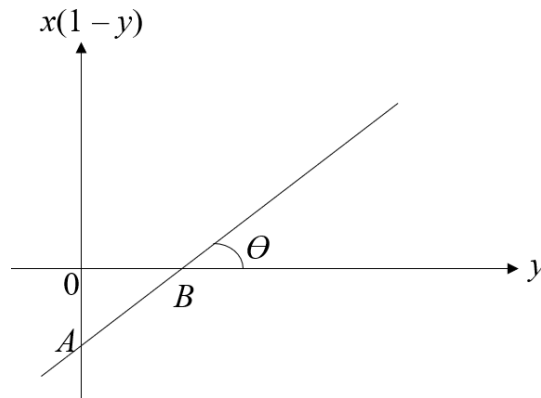
(b) $10\log_y 5 + 3 = \log_5 y$ [4]

a	$\log_2(x+4) = 2\log_2 x - 1$ $2\log_2 x - \log_2(x+4) = 1$ $\log_2 x^2 - \log_2(x+4) = 1$ $\log_2 \frac{x^2}{x+4} = 1$ $\frac{x^2}{x+4} = 2^1$ $x^2 - 2x - 8 = 0$ $(x-4)(x+2) = 0$ $x = 4, \quad x = -2 \text{ (rej)}$
b	$10\log_y 5 + 3 = \log_5 y$ $10\left(\frac{\log_5 5}{\log_5 y}\right) + 3 = \log_5 y$ $\frac{10}{\log_5 y} + 3 = \log_5 y$ <p>Let $\log_5 y = u$</p> $\frac{10}{u} + 3 = u$ $10 + 3u = u^2$ $u^2 - 3u - 10 = 0$ $(u-5)(u+2) = 0$ $u = 5, \quad u = -2$ $\log_5 y = 5 \quad \log_5 y = -2$ $y = 5^5 \quad y = 5^{-2}$ $y = 3125 \quad y = \frac{1}{25}$

- 3 The variables x and y are related by $y = \frac{2x+8}{2x+1}$. When values of $x(1-y)$ are plotted against y , a straight line is obtained. The straight line intersects the vertical and horizontal axes at A and B respectively.

(i) Find the coordinates of A and of B . [4]

(ii) State the value of $\tan \theta$. [1]



i	$y = \frac{2x+8}{2x+1}$ $2xy + y = 2x + 8$ $2x - 2xy = y - 8$ $2x(1-y) = y - 8$ $x(1-y) = \frac{1}{2}y - 4$ $A = (0, -4)$ $\text{sub } x(1-y) = 0,$ $0 = \frac{1}{2}y - 4$ $y = 8$ $B = (8, 0)$
ii	$\tan \theta = \frac{1}{2}$

4 (i) Factorise completely $2x^3 - 3x^2 - 5x + 6$.

[4]

(ii) Hence, solve $2e^{3y} - 3e^{2y} - 5e^y + 6 = 0$.

[3]

i	$f(x) = 2x^3 - 3x^2 - 5x + 6$ $f(1) = 2 - 3 - 5 + 6 = 0$ $(x-1) \text{ is a factor}$ $\begin{array}{r} 2x^2 - x - 6 \\ x-1 \overline{) 2x^3 - 3x^2 - 5x + 6} \\ \underline{-(2x^3 - 2x^2)} \\ -x^2 - 5x + 6 \\ \underline{-(-x^2 + x)} \\ -6x + 6 \\ \underline{-(-6x + 6)} \\ 0 \end{array}$ $2x^3 - 3x^2 - 5x + 6 = (x-1)(2x^2 - x - 6)$ $= (x-1)(2x+3)(x-2)$
ii	$2e^{3y} - 3e^{2y} - 5e^y + 6 = 0$ $e^y = 1, \quad e^y = -\frac{3}{2} \text{ (rej)}, \quad e^y = 2$ $y = 0, \quad y = \ln 2$

5 (i) Prove that $\frac{2 - \sec^2 \theta}{\sec \theta (\sin \theta + \cos \theta)} = 1 - \tan \theta$. [4]

(ii) Hence solve the equation $\frac{4 - 2\sec^2 \theta}{\sec \theta (\sin \theta + \cos \theta)} = \sec^2 \theta - 2$ for $0 \leq \theta \leq 2\pi$. [5]

i	$\begin{aligned} \text{LHS} &= \frac{2 - \sec^2 \theta}{\sec \theta (\sin \theta + \cos \theta)} \\ &= \frac{2 - (1 + \tan^2 \theta)}{\frac{1}{\cos \theta} (\sin \theta + \cos \theta)} \\ &= \frac{1 - \tan^2 \theta}{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\cos \theta}} \\ &= \frac{1 - \tan^2 \theta}{\tan \theta + 1} \\ &= \frac{(1 + \tan \theta)(1 - \tan \theta)}{\tan \theta + 1} \\ &= 1 - \tan \theta \\ &= \text{RHS} \end{aligned}$ <p>OR</p> $\begin{aligned} \text{LHS} &= \frac{2 - \sec^2 \theta}{\sec \theta (\sin \theta + \cos \theta)} \\ &= \left(2 - \frac{1}{\cos^2 \theta} \right) \div \left[\frac{1}{\cos \theta} (\sin \theta + \cos \theta) \right] \\ &= \frac{2\cos^2 \theta - 1}{\cos^2 \theta} \times \frac{\cos \theta}{\sin \theta + \cos \theta} \\ &= \frac{2\cos^2 \theta - 1}{\cos \theta (\sin \theta + \cos \theta)} \\ &= \frac{2\cos^2 \theta - (\sin^2 \theta + \cos^2 \theta)}{\cos \theta (\sin \theta + \cos \theta)} \\ &= \frac{\cos^2 \theta - \sin^2 \theta}{\cos \theta (\sin \theta + \cos \theta)} \\ &= \frac{(\cos \theta + \sin \theta)(\cos \theta - \sin \theta)}{\cos \theta (\sin \theta + \cos \theta)} \\ &= \frac{(\cos \theta + \sin \theta)(\cos \theta - \sin \theta)}{\cos \theta (\sin \theta + \cos \theta)} \\ &= \frac{\cos \theta - \sin \theta}{\cos \theta} \\ &= 1 - \frac{\sin \theta}{\cos \theta} \\ &= 1 - \tan \theta \\ &= \text{RHS} \end{aligned}$
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ii	$\frac{4 - 2 \sec^2 \theta}{\sec \theta (\sin \theta + \cos \theta)} = \sec^2 \theta - 2$ $2(1 - \tan \theta) = \sec^2 \theta - 2$ $2 - 2 \tan \theta = 1 + \tan^2 \theta - 2$ $\tan^2 \theta + 2 \tan \theta - 3 = 0$ $(\tan \theta + 3)(\tan \theta - 1) = 0$ $\tan \theta = -3, \quad \tan \theta = 1$ $\alpha = 1.2490 \quad \alpha = \frac{\pi}{4}$ $\theta = \pi - 1.2490, 2\pi - 1.2490 \quad \theta = \frac{\pi}{4}, \pi + \frac{\pi}{4}$ $\theta = 1.8925, 5.0341 \quad \theta = \frac{\pi}{4}, \frac{5\pi}{4}$ $\theta = 1.89, 5.03 \text{ (3sf)}$
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- 6 The first two non-zero terms in the expansion of $(1+bx)(1+ax)^6$ in ascending powers of x are 1 and $-\frac{21}{4}x^2$. Find the value of each of the constants a and b , where $a < b$. [7]

$$(1+ax)^6 = 1^6 + \binom{6}{1}(1^5)(ax) + \binom{6}{2}(1^4)(ax)^2 + \dots$$

$$= 1 + 6ax + 15a^2x^2 + \dots$$

$$(1+bx)(1+ax)^6 = (1+bx)(1 + 6ax + 15a^2x^2 + \dots)$$

$$= 1 + 6ax + bx + 15a^2x^2 + 6abx^2 + \dots$$

$$6a + b = 0$$

$$b = -6a \quad \text{---(1)}$$

$$15a^2 + 6ab = -\frac{21}{4} \quad \text{---(2)}$$

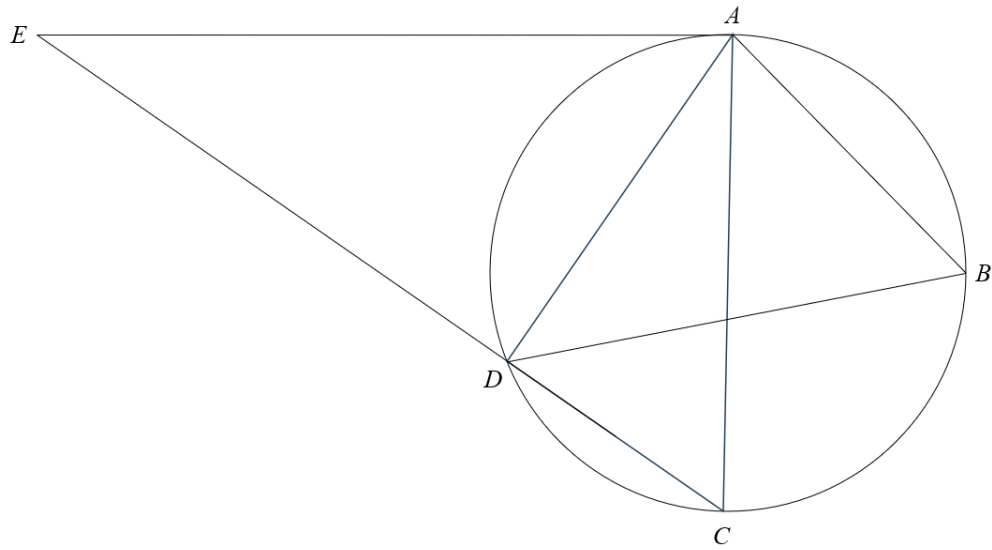
Sub (1) into (2):

$$15a^2 + 6a(-6a) = -\frac{21}{4}$$

$$a^2 = \frac{1}{4}$$

$$a = -\frac{1}{2}, \quad a = \frac{1}{2} \text{ (rej)}$$

$$b = 3, \quad b = -3 \text{ (rej)}$$



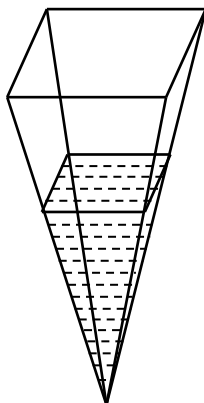
The diagram shows a circle passing through the points A , B , C and D . AC is a diameter of the circle. The line EA is a tangent to the circle and it intersects the straight line EDC at E .

(i) Show that angle $AED =$ angle DAC . [2]

(ii) Show that $AD^2 = CD \times DE$. [4]

i	<p>Let $\angle AED = \theta$</p> <p>$\angle EAD = 180 - 90 - \theta$ $= 90 - \theta$ (\angle in semicircle or sum of \angles in a Δ)</p> <p>$\angle DAC = 90 - (90 - \theta)$ $= \theta$ (tangent \perp radius) $= \angle AED$</p>
ii	<p>$\angle EAD = \angle ACD$ (\angles in alternate segment)</p> <p>$\angle ADE = \angle CDA = 90^\circ$ (\angle in semicircle)</p> <p>$\angle DEA = \angle DAC$ (sum of \angles in a Δ)</p> <p>$\triangle DEA$ similar to $\triangle DAC$ (AAA)</p> <p>$\frac{DE}{DA} = \frac{EA}{AC} = \frac{DA}{DC}$</p> <p>$\frac{DE}{DA} = \frac{DA}{DC}$</p> <p>$AD^2 = CD \times DE$</p> <p>OR</p> <p>$\angle DEA = \angle DAC$ (from i)</p> <p>$\angle ADE = \angle CDA = 90^\circ$ (\angle in semicircle)</p> <p>$\angle EAD = \angle ACD$ (sum of \angles in a Δ)</p> <p>$\triangle DEA$ similar to $\triangle DAC$ (AAA)</p>

- 8 A vessel in the shape of an inverted right pyramid has a square base of side 12 cm and a height of 30 cm. Water is leaking from the vessel at a constant rate of $5 \text{ cm}^3/\text{s}$.



- (i) Show that the volume of water in the vessel, $V \text{ cm}^3$, is given by $V = \frac{4h^3}{75}$, where h is the depth of the water. [2]
- (ii) Find the rate of change of the depth of water when the water is 6 cm deep. [3]

i	<p>Let x be the length of the side of the water surface.</p> $\frac{x}{12} = \frac{h}{30}$ $x = \frac{2h}{5}$ $V = \frac{1}{3} \left(\frac{2h}{5} \right)^2 (h)$ $V = \frac{4h^3}{75}$
ii	$V = \frac{4h^3}{75}$ $\frac{dV}{dh} = \frac{4}{25} h^2$ $\frac{dV}{dh} = \frac{dV}{dt} \div \frac{dh}{dt}$ $\frac{4}{25} (6)^2 = -5 \div \frac{dh}{dt}$ $\frac{dh}{dt} = -\frac{125}{144} \text{ cm/s}$

- 9 $f(x)$ is such that $f'(x) = \sin \frac{1}{4}x - \cos 4x$. Given that $f(2\pi) = 1$, show that

$$16f''(x) + f(x) = a \sin 4x + b, \text{ where } a \text{ and } b \text{ are constants.}$$

[6]

$$f(x) = -4 \cos \frac{1}{4}x - \frac{1}{4} \sin 4x + c$$

$$\text{Sub } f(2\pi) = 1, \quad -4 \cos \frac{2\pi}{4} - \frac{1}{4} \sin 8\pi + c = 1$$

$$c = 1$$

$$f(x) = -4 \cos \frac{1}{4}x - \frac{1}{4} \sin 4x + 1$$

$$f''(x) = \frac{1}{4} \cos \frac{1}{4}x + 4 \sin 4x$$

$$16f''(x) + f(x)$$

$$= 16\left(\frac{1}{4} \cos \frac{1}{4}x + 4 \sin 4x\right) - 4 \cos \frac{1}{4}x - \frac{1}{4} \sin 4x + 1$$

$$= 4 \cos \frac{1}{4}x + 64 \sin 4x - 4 \cos \frac{1}{4}x - \frac{1}{4} \sin 4x + \frac{1}{4}$$

$$= 63\frac{3}{4} \sin 4x + 1$$

- 10** **(a)** **(i)** Find the range of values of the constant m for which the curve $y = x^2 - 5x + m$ meets the line $y = mx - 8$. [4]
- (ii)** Hence state the values of m for which the line is a tangent. [1]
- (b)** Given that $px^2 + 5x - q$ is always positive, what conditions must apply to the constants p and q ? [3]

ai	$y = x^2 - 5x + m$ --- (1) $y = mx - 8$ --- (2) (1) = (2): $x^2 - 5x + m = mx - 8$ $x^2 - 5x - mx + m + 8 = 0$ $x^2 - (5 + m)x + m + 8 = 0$ $b^2 - 4ac \geq 0$ $[-(5 + m)]^2 - 4(1)(m + 8) \geq 0$ $25 + 10m + m^2 - 4m - 32 \geq 0$ $m^2 + 6m - 7 \geq 0$ $(m - 1)(m + 7) \geq 0$ $m \leq -7, \quad m \geq 1$
aii	$m = -7, \quad m = 1$
b	$px^2 + 5x - q$ always positive: $p > 0$ $5^2 - 4p(-q) < 0$ $25 + 4pq < 0$ $pq < -\frac{25}{4}$

- 11** (i) Express $\frac{-x^2+2x+1}{(x-1)(x^2+1)}$ in the form $\frac{A}{x-1} + \frac{Bx+C}{x^2+1}$ where A , B and C are constants.

[4]

- (ii) Differentiate $\ln(x^2+1)$ with respect to x .

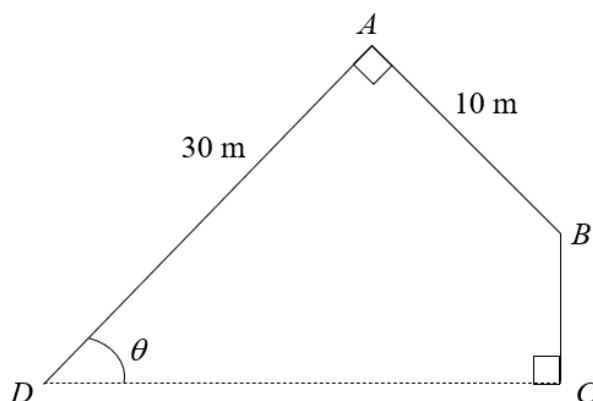
[1]

- (iii) Using your results from parts (i) and (ii), find $\int \frac{-2x^2+4x+2}{(x-1)(x^2+1)} dx$.

[2]

i	$\frac{-x^2+2x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$ $-x^2+2x+1 = A(x^2+1) + (Bx+C)(x-1)$ <p>Sub $x=1$,</p> $2 = A(2) + 0$ $A = 1$ <p>Sub $x=0$,</p> $1 = 1 + C(-1)$ $C = 0$ <p>Sub $x=2$,</p> $1 = 5 + (2B)(1)$ $B = -2$ $\frac{-x^2+2x+1}{(x-1)(x^2+1)} = \frac{1}{x-1} - \frac{2x}{x^2+1}$
ii	$\frac{d}{dx} \ln(x^2+1) = \frac{2x}{x^2+1}$
iii	$\int \frac{-2x^2+4x+2}{(x-1)(x^2+1)} dx = 2 \int \frac{1}{x-1} - \frac{2x}{x^2+1} dx$ $= 2[\ln(x-1) - \ln(x^2+1)] + c$

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The diagram shows an area that is enclosed by glass panels at AB , BC and AD . $AB = 10$ m, $AD = 30$ m, angle $DAB = \text{angle } BCD = 90^\circ$. The glass panel AD makes an acute angle θ with CD .

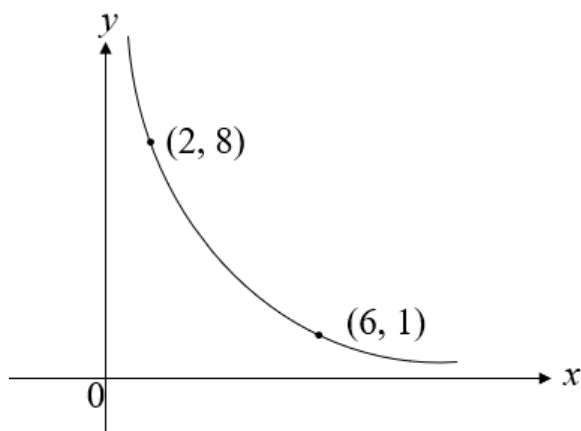
- (i) Show that L m, the length of the glass panels, can be expressed as $40 + 30\sin\theta - 10\cos\theta$. [2]
- (ii) Express L in the form $40 + R\sin(\theta - \alpha)$ where $R > 0$ and $0^\circ < \alpha < 90^\circ$. [4]
- (iii) The total length of the glass panels is 65 m. Find the value of θ . [2]
- (iv) Explain whether it is possible to build an area where the length of the glass panels is 90 m. [1]

i	<p> $\sin\theta = \frac{AF}{30}$ $AF = 30\sin\theta$ </p> <p> $\cos\theta = \frac{AE}{10}$ $AE = 10\cos\theta$ </p> <p> $L = 30 + 10 + 30\sin\theta - 10\cos\theta$ $L = 40 + 30\sin\theta - 10\cos\theta$ </p>
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[Turn over

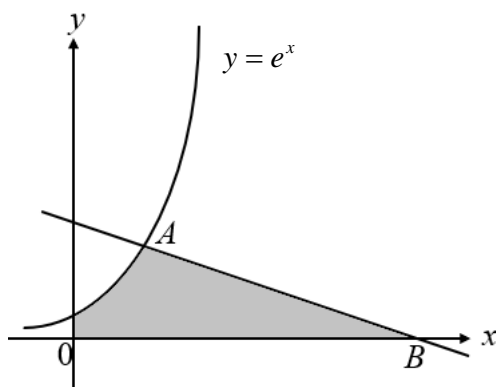
ii	$R = \sqrt{30^2 + 10^2}$ $R = \sqrt{1000}$ $R = 10\sqrt{10}$ $\alpha = \tan^{-1} \frac{10}{30}$ $\alpha = 18.434$ $L = 40 + 10\sqrt{10} \sin(\theta - 18.434)$ $L = 40 + 10\sqrt{10} \sin(\theta - 18.4) \text{ (1dp)}$
iii	$40 + 10\sqrt{10} \sin(\theta - 18.434) = 65$ $\sin(\theta - 18.434) = \frac{25}{10\sqrt{10}}$ $\alpha = 52.238$ $\theta - 18.434 = 52.238$ $\theta = 70.672$ $\theta = 70.7^\circ \text{ (1dp)}$
iv	<p>Since the maximum value of $40 + 10\sqrt{10} \sin(\theta - 18.434) = 40 + 10\sqrt{10} < 90$, it is <u>not possible</u> to build the area.</p> <p>OR</p> $40 + 10\sqrt{10} \sin(\theta - 18.434) = 90$ $\sin(\theta - 18.434) = 1.5811$ <p>Since $\sin(\theta - 18.434) \leq 1$, it is <u>not possible</u> to build the area.</p>

- 13 (a) The figure shows part of the curve $y = g(x)$. $(2, 8)$ and $(6, 1)$ are two points on the curve.



Given that $\int_2^6 y \, dx = 32$, find the value of $\int_1^8 x \, dy$. [2]

(b)



The diagram shows part of the curve $y = e^x$. The normal to the curve at point A where $x = 1$ intersects the x -axis at point B. Find the area of the shaded region.

Leave your answer in **exact form**. [7]

i	$\int_2^8 x \, dy = 32 - (4 \times 1) + (7 \times 2)$ $= 42$
ii	$y = e^x$ $\frac{dy}{dx} = e^x$ $m_{normal} = -\frac{1}{e}$ <p>Subs $(1, e), m_{normal} = -\frac{1}{e},$</p> $e = -\frac{1}{e}(1) + c$ $c = e + \frac{1}{e}$ <p>Normal: $y = -\frac{1}{e}x + e + \frac{1}{e}$</p> <p>Sub $y = 0,$</p> $-\frac{1}{e}x + e + \frac{1}{e} = 0$ $\frac{1}{e}x = e + \frac{1}{e}$ $x = e^2 + 1$ $\int_0^1 e^x \, dx = [e^x]_0^1$ $= e - 1$ <p>Area of triangle $= \frac{1}{2}(e^2 + 1 - 1)(e)$</p> $= \frac{1}{2}e^3$ <p>Area of region $= \frac{1}{2}e^3 + e - 1$</p>