CEDAR GIRLS' SECONDARY SCHOOL
Preliminary Examination
Secondary Four

CANDIDATE NAME	WORKED SOLUTIONS								
CLASS	4				INDEX NUMBER				
CENTRE/ INDEX NO					/				

ADDITIONAL MATHEMATICS
Paper 2

4049/02
26 August 2024

2 hours 15 minutes

Candidates answer on the Question Paper.

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use
90

This document consists of **21** printed pages and **1** blank page.

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Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ \ldots\ (n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

Answer **all** the questions.

1 **(a)** The curve $y = mx^2 - 12x + 2\,(x^2 + m - 1)$ lies entirely below the x-axis for all real values of x. Find the largest integer value of m. [5]

$y = mx^2 - 12x + 2\,(x^2 + m - 1)$
$\;\;= mx^2 - 12x + 2x^2 + 2m - 2$
$\;\;= (m+2)x^2 - 12x + 2m - 2$

Since the curve lies entirely below the x-axis,

① Discriminant, $b^2 - 4ac < 0$

$$(-12)^2 - 4(m+2)(2m-2) < 0$$
$$144 - 4(2m^2 + 2m - 4) < 0$$
$$-8m^2 - 8m + 160 < 0$$
$$m^2 + m - 20 > 0$$
$$(m+5)(m-4) > 0$$
$$m < -5 \quad \text{or} \quad m > 4$$

② Coefficient of x^2, $m + 2 < 0 \Rightarrow m < -2$

Combining the inequalities, $m < -5$

Largest integer value of $m = -6$

(b) Show that the roots of the equation $2x^2 - 3(1-x) = -p$ are real if $p \le 4\frac{1}{8}$. [4]

$2x^2 - 3(1-x) = -p \Rightarrow 2x^2 + 3x + p - 3 = 0$

$b^2 - 4ac = 9 - 8(p-3) = 33 - 8p$

Since $p \le 4\frac{1}{8}$, $p \le \frac{33}{8}$, therefore $33 - 8p \ge 0$

$b^2 - 4ac = 9 - 8(p-3) = 33 - 8p \ge 0$

Hence the roots are real.

2 Let $f(x) = \frac{3-3x^2}{(2x+1)(x+2)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{3-3x^2}{(2x+1)(x+2)^2} = \frac{A}{2x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$3-3x^2 = A(x+2)^2 + B(2x+1)(x+2) + C(2x+1)$

Let $x=-2$, $-9=-3C \Rightarrow C=3$

Let $x=-\frac{1}{2}$, $\frac{9}{4}=\frac{9}{4}A \Rightarrow A=1$

Comparing coefficients of x^2, $-3 = A+2B \Rightarrow B=-2$

$$\frac{3-3x^2}{(2x+1)(x+2)^2} = \frac{1}{2x+1} - \frac{2}{x+2} + \frac{3}{(x+2)^2}$$

(b) Hence find the value of $\int_0^4 \mathrm{f}(x)\,\mathrm{d}x$, giving your answer in the form $a+b\ln c$, where *a*, *b* and *c* are integers. [5]

$$\int_0^4 \mathrm{f}(x)\,\mathrm{d}x = \int_0^4\left(\frac{1}{2x+1}-\frac{2}{x+2}+\frac{3}{(x+2)^2}\right)\mathrm{d}x$$

$$=\left[\frac{\ln(2x+1)}{2}-2\ln(x+2)-\frac{3}{(x+2)}\right]_0^4$$

$$=\left[\frac{\ln 9}{2}-2\ln(6)-\frac{3}{(6)}\right]-\left[\frac{\ln 1}{2}-2\ln 2-\frac{3}{2}\right]$$

$$=\frac{2\ln 3}{2}-2\ln 6+2\ln 2+1$$

$$=1+\ln\frac{3(4)}{36}$$

$$=1+\ln\frac{1}{3}$$

$$=1+\ln 1-\ln 3=1-\ln 3$$

3 **(a)** **(i)** Prove $\operatorname{cosec} 2\theta - \cot 2\theta = \tan\theta$. [3]

$$LHS = \frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta}$$
$$= \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$
$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

(ii) Hence, solve $\operatorname{cosec} 4\theta - \cot 4\theta = -\sqrt{3}$ for $0 < \theta < \pi$. [2]

Hence $\tan 2\theta = -\sqrt{3}$ [Making use of the identity]

Basic angle = $\frac{\pi}{3} = 1.0472$

2θ lies in the 2nd or 4th quadrant and $0 < 2\theta < 2\pi$.

$2\theta = \pi - \frac{\pi}{3}$ or $2\theta = 2\pi - \frac{\pi}{3}$

$\theta = \frac{\pi}{3} = 1.05$ or $\theta = \frac{5\pi}{6} = 2.62$

(b) The angles A and B are such that

$$\sin(A+45°)=\left(2\sqrt{2}\right)\cos A \quad \text{and} \quad 4\sec^2 B+5=12\tan B.$$

Without using a calculator, find the exact value of $\tan(A-B)$. [5]

$\sin A\cos 45°+\cos A\sin 45°=(2\sqrt{2})\cos A$

$\frac{\sqrt{2}}{2}\sin A+\frac{\sqrt{2}}{2}\cos A=(2\sqrt{2})\cos A$ [Use of addition formula]

$\frac{\sqrt{2}}{2}\sin A+=\frac{3\sqrt{2}}{2}\cos A$

$\tan A=\frac{3\sqrt{2}}{2}\times\frac{2}{\sqrt{2}}=3$

$4(1+\tan^2 B)+5=12\tan B$

$4\tan^2 B-12\tan B+9=0$ [Use of $\sec^2 B=1+\tan^2 B$ and form a QE.]

$\left(2\tan B-3\right)^2=0$

$\tan B=\frac{3}{2}$

$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

$=\frac{3-\frac{3}{2}}{1+3\left(\frac{3}{2}\right)}=\frac{3}{11}$

4 The fifth term in the expansion of $\left(px-\dfrac{q}{x}\right)^n$, where p and q are positive numbers, is independent of x.

(a) Show that $n = 8$. [2]

Since it is fifth term, $T_5 = T_{4+1} = \binom{n}{4}(px)^{n-4}\left(-\dfrac{q}{x}\right)^4$ [for $r = 4$]

and it is independent, powers of $x =$ $n-4-4=0$

Therefore $n = 8$.

(b) Hence, explain why the fifth term is a positive constant. [1]

Since p and q are positive, $(p)^4 > 0$ and $(-q)^4 > 0$, the fifth term is a positive constant.

It is given that $p = 3$ and $q = 1$.

(c) Find the term independent of x in $\left(2+\frac{1}{x^2}\right)\left(px-\frac{q}{x}\right)^n$ for n = 8. [4]

$$\left(2+\frac{1}{x^2}\right)\left(3x-\frac{1}{x}\right)^8 = \left(2+\frac{1}{x^2}\right)\left(\ldots + \text{Independent term + Term in } x^2 + \ldots\right)$$

General term of $\left(3x-\frac{1}{x}\right)^8 = \binom{8}{r}(3x)^{8-r}\left(-\frac{1}{x}\right)^r$

For independent term, $8-2r=0$

$r=4$

Independent term = $\binom{8}{4}(3)^4(-1)^4 = 5670$

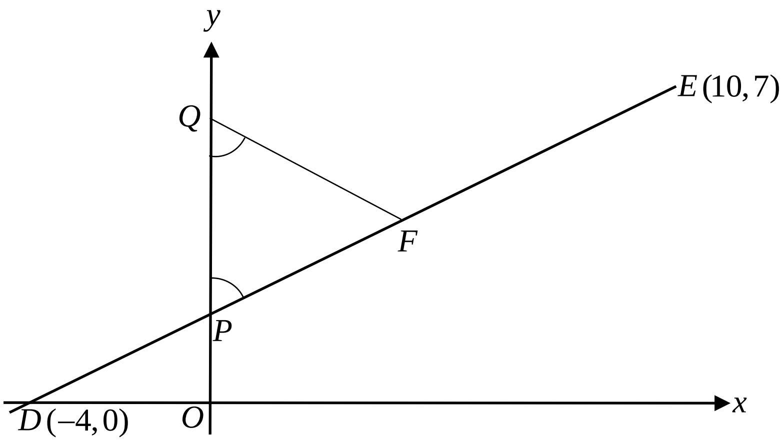
For term in x^2, $8-2r=2$

$r=3$

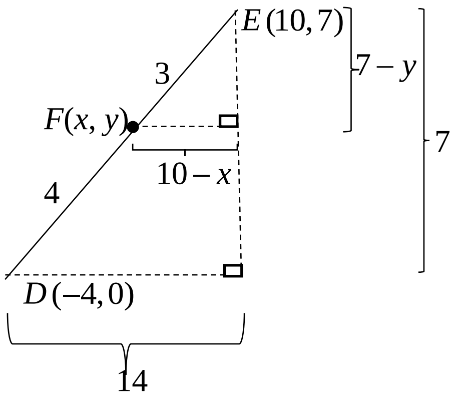
Term in $x^2 = \binom{8}{3}(3)^5(-1)^3 = -13608x^2$

Independent term in $\left(2+\frac{1}{x^2}\right)\left(3x-\frac{1}{x}\right)^8 = 2(5670)-13608 = -2268$

5 The diagram shows a line DE which cuts the y-axis at P and a line through F meets the y-axis at Q such that angle $FPQ =$ angle FQP.
The coordinates of D and E are $(-4, 0)$ and $(10, 7)$ respectively.
F is a point on the line DE such that $DF : FE = 4 : 3$.



(a) Find the coordinates of F. [3]



$$\boxed{\frac{7-y}{7} = \frac{3}{7} \Rightarrow y = 4}$$ M1

$$\boxed{\frac{10-x}{14} = \frac{3}{7} \Rightarrow x = 4}$$ M1

Coordinates of F = (4, 4)

(b) Find the equation of the straight line FQ. [2]

Gradient of FQ = $-\frac{1}{2}$

Equation of FQ : $y-4=-\frac{1}{2}(x-4)$

$$y=-\frac{1}{2}x+6$$

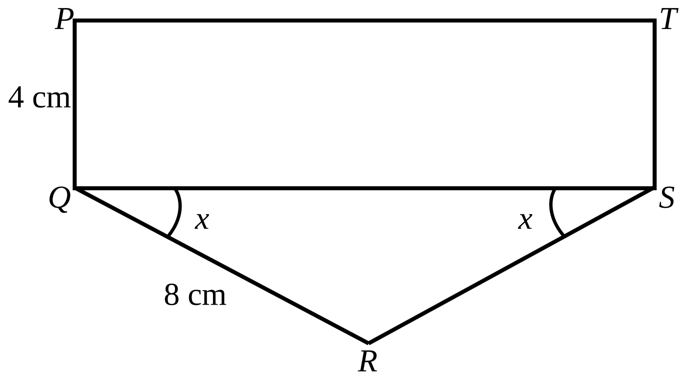
(c) Find the area of the triangle QFE. [3]

When x = 0, y = 6, Coordinates of Q = $(0,6)$

Area of triangle QFE = $\frac{1}{2}\begin{vmatrix} 0 & 4 & 10 & 0 \\ 6 & 4 & 7 & 6 \end{vmatrix}$

= 12 sq units

6



The diagram shows a figure *PQRST* which consists of a rectangle *PQST* and an isosceles triangle *QRS*.
It is given that *PQ* = 4 cm and *QR* = 8 cm.

(a) Given angle $SQR =$ angle $QSR = x$ radians and the area of *PQRST* is given by A cm^2, show that $A = 64\cos x + 32\sin 2x$. [4]

$$\cos x = \frac{\frac{1}{2}QS}{8}$$

$$\Rightarrow QS = 16\cos x$$

$$\text{Area of } \Delta QRS = \frac{1}{2}\times 8\times(16\cos x)\sin x$$

$$= 64\sin x\cos x \text{ cm}^2$$

$$\text{Area of } \square PQST = 4\times 16\cos x$$

$$= 64\cos x \text{ cm}^2$$

$$\therefore A = \text{area of } \Delta QRS + \text{area of } \square PQST$$

$$= 64\cos x + 64\sin x\cos x$$

$$= 64\cos x + 32\sin 2x \quad \text{(shown)}$$

(b) Find the value of x for which A has a stationary value. [3]

$A = 64\cos x + 32\sin 2x$

$\frac{\mathrm{d}A}{\mathrm{d}x} = -64\sin x + 64\cos 2x$

For stationary value, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$

$-64\sin x + 64\cos 2x = 0$

$\cos 2x - \sin x = 0$

$1 - 2\sin^2 x - \sin x = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}$ or $\sin x = -1$ (rejected)

$x = \frac{\pi}{6} = 0.524$

(c) Hence find the exact stationary value of A and determine whether it is a maximum or a minimum. [2]

$$A = 64\cos\frac{\pi}{6} + 32\sin 2\left(\frac{\pi}{6}\right)$$

$$= 64\left(\frac{\sqrt{3}}{2}\right) + 32\left(\frac{\sqrt{3}}{2}\right)$$

$$= 32\sqrt{3} + 16\sqrt{3}$$

$$= 48\sqrt{3}\ \text{cm}^2$$

$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -64\cos x - 128\sin 2x$

when $x = \frac{\pi}{6}$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} < 0 \Rightarrow A$ has a maximum value.

7 It is given that $f(x) = x^3 + ax^2 - 5x + b$, where a and b are constants, has a factor of $x+1$ and leaves a remainder of -24 when divided by $(x+3)$.

(a) Show that $a = -1$ and $b = -3$. [4]

Since x + 1 is a factor, $f(-1) = 0$

$$f(-1) = (-1)^3 + a(-1)^2 - 5(-1) + b = 0$$

$$a + b = -4$$

Since R = -24 when divided by $(x+3)$, $f(-3) = -24$

$$f(-3) = (-3)^3 + a(-3)^2 - 5(-3) + b = -24$$

$$9a + b = -12$$

$a = -1$ and $b = -3$.

(b) Hence factorise f (x) completely and write down the solutions for $\mathrm{f}(x)=0$. [3]

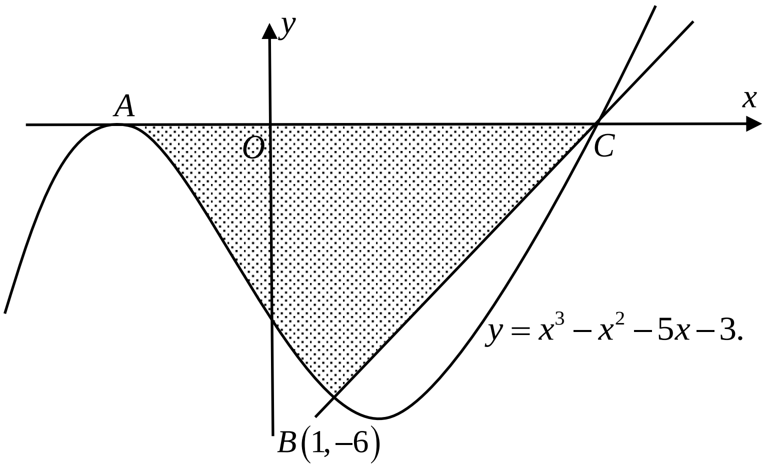
$$x+1 \overline{\left) x^3-x^2-5x-3. \right.}^{\;x^2-2x-3}$$

$\mathrm{f}(x)=(x+1)^2(x-3)$

$\mathrm{f}(x)=(x+1)^2(x-3)=0$

$x=-1$ or $x=3$

From part **(a)**, it is known that $f(x)=x^3-x^2-5x-3$.
The diagram below shows part of the curve $y = f(x)$ and a line that cuts the curve at B and C. B has coordinates $(1,-6)$.
A and C are points on the curve and the x-axis.



(c) Find the area of the shaded region. [5]

$$\left.\begin{array}{l} A=(-1,0) \\ C=(3,0) \end{array}\right\} \text{Either one}$$

Area of shaded region $=\left|\int_{-1}^{1}\left(x^3-x^2-5x-3\right)\mathrm{d}x\right|+\frac{1}{2}\times(3-1)\times 6$

[Area = Integration under curve + Area of triangle]

$$=\left|\left[\frac{x^4}{4}-\frac{x^3}{3}-\frac{5x^2}{2}-3x\right]_{-1}^{1}\right|+6$$

$$=\left|\frac{(1)^4}{4}-\frac{(1)^3}{3}-\frac{5(1)^2}{2}-3(1)-\left(\frac{(-1)^4}{4}-\frac{(-1)^3}{3}-\frac{5(-1)^2}{2}-3(-1)\right)\right|+6$$

$=\left|-\frac{2}{3}-6\right|+6=12\frac{2}{3}$ sq units [Final Area]

(d) Show that the equation of the tangent at the minimum point of the curve is $y = -\frac{256}{27}$. [3]

$f'(x) = 3x^2 - 2x - 5$

For minimum point, $3x^2 - 2x - 5 = 0$

$(3x-5)(x+1) = 0$

$x = \frac{5}{3}$ or $x = -1$

Equation of the tangent: $y = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) - 3 = -\frac{256}{27}$

8 A circle has a diameter AB. The point A has coordinates (1, − 6) and the equation of the tangent to the circle at B is $3x+4y=k$.

(a) Show that the equation of the normal to the circle at the point A is $4x-3y=22$. [3]

Since the normal at A will pass through the centre of the circle and ultimately B, it will be perpendicular to the tangent at B.

$$3x+4y=k \Rightarrow y=-\frac{3x}{4}+\frac{k}{4}$$

Gradient of normal at A = $\frac{-1}{-3/4}=\frac{4}{3}$ [Attempt to find gradient of normal]

Equation of normal at A: $y-(-6)=\frac{4}{3}(x-1) \Rightarrow y+6=\frac{4}{3}x-\frac{4}{3}$

$$4x-3y=22$$

It is also given that the line $x=-1$ touches the circle at the point (–1, –2).

(b) Find the coordinates of the centre and the radius of the circle. [4]

Since the line x = –1 touches the circle at the point (–1, –2), so the equation of the normal at (–1, –2) is y = –2 and this passes through the centre of the circle. Solving the equations $4x-3y=22$ and y = –2 provides the x coordinate of centre of circle and the y coordinate of centre of circle is -2 .

$4x-3(-2)=22 \Rightarrow 4x=16 \Rightarrow x=4$

Coordinates of the centre = $(4,-2)$

Radius of circle = $\sqrt{(4-1)^2+(-6-(-2))^2}$

= 5 units

Or Radius of circle (from (–1, –2) to $(4,-2)$ = 1+ 4= 5 units

Continuation of working space for question **8 (b).**

(c) Find the value of *k*. [3]

Let the coordinates of *B* be (a, b).

$$\left(\frac{a+1}{2},\frac{b-6}{2}\right)=\left(4,-2\right)$$

$a = 2(4)–1= 7$ and $b = 2(–2) + 6 = 2$

Coordinates of $B = (7,2)$,

Sub. (7, 2) into $3x + 4y = k$,

$k = 3(7) + 4(2) = 29$

9 Mr Chan was driving a car along a straight road. He was 35 m away from the stop line when he applied his brakes near a traffic light. His acceleration, a m/s^2, after applying the brakes was given by $a = -7.5e^{-\frac{t}{2}}$ where t is the time in seconds after he applied the brakes.

(a) Explain mathematically why $a < 0$ for all $t \geq 0$ and the significance of $a < 0$. [2]

As $e^{-\frac{t}{2}} > 0$ for all $t \geq 0$, then $a = -7.5e^{-\frac{t}{2}} < 0$. [Both concepts]
This means that the car is decelerating.

(b) Mr Chan's car was travelling at 14 m/s just before he applied his brakes.
Express the velocity of his car, v m/s, in terms of t. [3]

$$v = \frac{-7.5e^{-\frac{t}{2}}}{-\frac{1}{2}} + c$$

$$v = 15e^{-\frac{t}{2}} + c$$

When $t = 0$, $v = 14$

$14 = 15(1) + c \Rightarrow c = -1$

$$v = 15e^{-\frac{t}{2}} - 1$$

(c) Hence find the time taken for his car to come to a complete stop. [2]

When it comes to a complete stop, $v = 0$.

$$15e^{-\frac{t}{2}} - 1 = 0$$
$$t = -2\ln\frac{1}{15}$$
$$t = -2\ln\frac{1}{15} = 5.4216 = 5.42$$

Time taken = 5.42 s

(d) Obtain an expression, in terms of t, for the displacement of Mr Chan's car from the point he applied the brakes. [3]

$$s = \int v \, \mathrm{d}t = -30e^{-\frac{t}{2}} - t + c$$

When $t = 0$, $s = 0$, $0 = -30(1) - 0 + c \Rightarrow c = 30$

$$s = -30e^{-\frac{t}{2}} - t + 30$$

(e) Determine if Mr Chan's car was able to come to a complete stop before reaching the stop line. Explain your answer. [2]

When $t = 5.4161$, $s = -30e^{-\frac{5.4161}{2}} - 5.4161 + 30 = 22.6 < 35$

Yes, his car came to a complete stop before reaching the stop line.

End of Paper

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